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Contents

Preface

The main objective of this thesis is the investigation of different models arising from mathematical biology and fluid mechanics in the time-periodic setting. We consider the *classical Keller–Segel model* for chemotaxis as well as its coupling to a fluid whose motion is described by the *Navier–Stokes equations*. The second model we investigate is the *bidomain system* which describes the propagation of electrophysiological waves in the heart. The last model considered is the *Beris–Edwards model* of nematic liquid crystals.

Chemotaxis

Chemotaxis describes the influence of chemical gradients on the movement of cells and organism. As an example, bacteria often swim towards a higher oxygen concentration to survive. Introduced by Keller and Segel in 1970 [52] the Keller–Segel model in its various forms became a prototype model describing chemotaxis. We will focus on the *classical Keller–Segel* system which reads as follows

$$
\text{(KS)} \qquad \begin{cases} \partial_t n = \Delta n - \nabla \cdot (n \nabla c) & \text{in } (0,\infty) \times \Omega, \\ \partial_t c = \Delta c - c + n & \text{in } (0,\infty) \times \Omega, \\ \dfrac{\partial n}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ (n,c)(x,0) = (n_0,c_0)(x) & \text{in } \Omega. \end{cases}
$$

Here, $\Omega \subset \mathbb{R}^d$ is the physical domain. The unknown functions n and c denote the density of cells or organisms and the concentration of a chemical attractant, respectively. The term $\nabla \cdot (n\nabla c)$ describes the aggregation of the cells towards higher concentration of the chemical signal. Furthermore, $-c$ characterizes that the activity of the signal decays with less concentration whereas $+n$ describes that a higher aggregation of cells enhances the signal production. Due to the latter, this type of Keller–Segel model describes a signal production mechanism.

For more details and other types of Keller–Segel systems we refer to the survey articles [45, 46, 59] and the references therein. Considering bounded smooth domains there are many results concerning local and global well-posedness as well as blow-up of solutions, see again the survey articles mentioned above.

Looking for periodic solutions or for results on non-smooth domains, the situation is quite different. On bounded smooth domains the existence and uniqueness of strong time-periodic solutions to (KS) was shown recently in [44]. See also the references therein for other results on time-periodic solutions to different kinds of Keller–Segel models, but note that neither of them uses the notion of strong solutions. For non-smooth domains we refer to [47] where local well-posedness for the *full Keller–Segel model*, a quasilinear strongly coupled reaction-crossdiffusion system of four parabolic equations, is shown for Lipschitz domains.

The classical Keller–Segel model describes chemotaxis without other outside influences. But in nature cells or bacteria often live in a viscous fluid, so the cells and chemical substrates are also transported by the fluid. Meanwhile, the motion of the fluid is under the influence of gravitational forcing generated by aggregation of cells.

Hence, it seems natural and interesting to not only consider the interaction of cells and chemical gradients via diffusion and chemotaxis but also include transport and viscous fluid dynamics. The motion of a viscous fluid is usually described by the viscous incompressible Navier–Stokes equations. For more information and known results concerning the Stokes equations we refer to the survey article from Hieber and Saal [43] and the references therein. For the Navier–Stokes equations we refer to [14, 26, 30] and the references therein. In order to describe the coupled biological phenomena, Tuval et al. [79] introduced for the first time a model which is a coupled system of the Keller–Segel model for chemotaxis and the

Navier–Stokes equations for viscous incompressible fluids.

We will consider a coupled chemotaxis-Navier–Stokes system of the form

$$\text{(KSNS)} \qquad \begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) & \text{in } (0,\infty) \times \Omega, \\ \partial_t c + u \cdot \nabla c = \Delta c - c + n & \text{in } (0,\infty) \times \Omega, \\ \partial_t u - \Delta u - \nabla P = \kappa (u \cdot \nabla) u + n \nabla \phi & \text{in } (0,\infty) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0,\infty) \times \Omega, \\ \dfrac{\partial n}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ (n,c,u)(x,0) = (n_0, c_0, u_0)(x) & \text{in } \Omega. \end{cases}$$

Here u and P denote the fluid velocity and the associated pressure, respectively. The transport of the cells and chemical substrates is described by $u \cdot \nabla n$ and $u \cdot \nabla c$ whereas $n \nabla \phi$ includes the gravitational forcing. Moreover, $\kappa \in \mathbb{R}$ is a fixed number which distinguishes between the Stokes and the Navier–Stokes case.

After establishing the first coupled chemotaxis fluid model in [79] there has been a rising interest in those kind of models over the last years. Since this model used a Keller–Segel model with signal consumption, the first mathematical results for chemotaxis-fluid systems also considered this type.

In [62] the existence of a local weak solution to this system was shown in bounded smooth domains. Later, for smoothly bounded convex domains in [82] the existence and uniqueness of a global classical solution in two space dimensions was obtained. Furthermore, for the simplified chemotaxis-Stokes system the existence of a global weak solution was shown in the 3D-case. This result was extended to the full chemotaxis-Navier–Stokes system in [83].

For systems with signal production mechanism there are less results available. In [63], they considered the coupling of an elliptic-parabolic Keller–Segel system with the Stokes equations in the whole space and showed the existence of a global weak solution provided the initial data satisfies some smallness conditions. Furthermore, they showed that the coupling with a fluid somehow delays the blow-up behavior of the Keller–Segel model without fluid. First results for a classical parabolic-parabolic

Keller–Segel model with signal production and coupled to the Navier–Stokes equations were obtained in [54]. They obtained global solutions in the mild sense in the whole space for small initial data. For a system on bounded smooth domains with singular sensitivity in [12] the global existence of classical solutions was shown, in 2D for Navier–Stokes fluids and in 3D for Stokes-fluids.

There are only few works dealing with the question of time-periodic solutions to chemotaxis-fluid systems. In [48, 49], they considered strong periodic solutions to a chemotaxis-Navier–Stokes (in 2D) and to a chemotaxis-Stokes system (in 3D) where the Keller–Segel part has an additional logistic-type term. The existence of time-periodic solutions is shown without assuming smallness for the external forces, but there is no uniqueness for these solutions.

Overview

It is the first main goal of this thesis to show the existence and uniqueness of strong time-periodic solutions to the periodic chemotaxis-Navier–Stokes system on bounded smooth domains

(PKSNS)
$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) + f_1(t) & \text{in } (0,\infty) \times \Omega, \\ \partial_t c + u \cdot \nabla c = \Delta c - c + n + f_2(t) & \text{in } (0,\infty) \times \Omega, \\ \partial_t u - \Delta u - \nabla P = \kappa (u \cdot \nabla) u + n \nabla \phi + f_3(t) & \text{in } (0,\infty) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0,\infty) \times \Omega, \\ \dfrac{\partial n}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ (n,c,u)(x,0) = (n,c,u)(x,T) & \text{in } \Omega. \end{cases}$$

Here the T-periodic external forces f_1, f_2, and f_3 are supposed to be sufficiently small. Our investigation will be done in two different settings, the *strong setting* and the *weak setting*, which should not be mixed up with weak solutions. The strong setting corresponds to a data space of the form

$$\mathbb{F} := L^p(0,T; L^q_{av}(\Omega) \times W^{1,q}(\Omega) \times L^q_\sigma(\Omega))$$

and to solution spaces

$$\begin{aligned}
\mathbb{E}_1 &:= L^p(0,T; W_N^{2,q}(\Omega) \cap L^q_{av}(\Omega)) \cap W^{1,p}(0,T; L^q_{av}(\Omega)),\\
\mathbb{E}_2 &:= L^p(0,T; W_N^{3,q}(\Omega)) \cap W^{1,p}(0,T; W^{1,q}(\Omega)),\\
\mathbb{E}_3 &:= L^p(0,T; W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega)) \cap W^{1,p}(0,T; L^q_\sigma(\Omega)),\\
\mathbb{E} &:= \mathbb{E}_1 \times \mathbb{E}_2 \times \mathbb{E}_3.
\end{aligned}$$

In order to cover also solutions, where the bacterial density n and the oxygen concentration c are nonnegative, we first consider a slightly modified model which is given by

(PKSNS II)

$$\begin{cases}
\partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot ((n+M)\nabla c) & \text{in } (0,\infty)\times\Omega,\\
\partial_t c + u \cdot \nabla c = \Delta c - c + n + f_2(t) & \text{in } (0,\infty)\times\Omega,\\
\partial_t u - \Delta u - \nabla P = \kappa(u\cdot\nabla)u + (n+M)\nabla\phi + f_3(t) & \text{in } (0,\infty)\times\Omega,\\
\nabla \cdot u = 0 & \text{in } (0,\infty)\times\Omega,\\
\dfrac{\partial n}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = 0 & \text{on } (0,\infty)\times\partial\Omega,\\
u = 0 & \text{on } (0,\infty)\times\partial\Omega,\\
(n,c,u)(x,0) = (n,c,u)(x,T) & \text{in } \Omega,
\end{cases}$$

for some $M \geq 0$. We rewrite this model as an abstract evolution equation. Then, we show by using maximal periodic regularity of the involved operator matrix and certain Lipschitz estimates for the nonlinear terms that the modified model admits a unique strong time-periodic solution. Due to the nonlinear estimates we need some conditions on p and q which depend on the considered setting. Note that this solution might be negative but it can be used to show the existence of nonnegative solutions to the original model (PKSNS). This is done in Theorem 3.1.1 and Corollary 3.1.2 for the strong setting.

Theorem. *Let $p, q \in (1,\infty)$ satisfy $1/p + d/(2q) < 1$. Let $M > 0$, $T > 0$ and assume that $f = (0, f_2, f_3)^T \in \mathbb{F}$ is T-periodic.*

a) Then there are $r_0 > 0$ and $M_0 > 0$ such that for any $r \in (0, r_0)$ there exists $\delta = \delta(r) > 0$ such that if $\|f\|_{\mathbb{F}} < \delta$ and $M < M_0$, then there exists a T-periodic solution $w = (n,c,u)^T \in \mathbb{E}$ to (PKSNS II)*, which is unique in $\overline{B_{\mathbb{E}}}(0,r)$.*

b) If in addition f_2 is nonnegative, then $(n+M, c+M, u) \in \mathbb{E}$ is a T-periodic solution with $(n+M, c+M)$ nonnegative to (PKSNS) *with $f_1 \equiv 0$.*

For the weak setting we obtain similar results in Theorem 3.1.4 and Corollary 3.1.5.

Afterwards we consider a quasilinear version of (PKSNS), where the term Δn is replaced by $\nabla \cdot ((n+1)^m)\nabla n)$ for some $m \in \mathbb{R}$. Using a similar strategy as before we show that the quasilinear system also has time-periodic solutions.

In Section 3.2 we consider the initial value problem (KSNS) in the time-weighted strong and weak setting. By applying the classical theory for quasilinear abstract parabolic evolution equations we show local well-posedness.

The next main objective are periodic solutions to the classic Keller–Segel model

$$\text{(PKS)} \qquad \begin{cases} \partial_t n = \Delta n - \nabla \cdot (n \nabla c) + f_1(t) & \text{in } \mathbb{R} \times \Omega, \\ \partial_t c = \Delta c - c + n + f_2(t) & \text{in } \mathbb{R} \times \Omega, \\ \dfrac{\partial n}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ n(x,0) = n(x,T) & \text{in } \Omega, \\ c(x,0) = c(x,T) & \text{in } \Omega, \end{cases}$$

on nonsmooth domains. To be more precise, we consider domains $\Omega \subset \mathbb{R}^3$ which are bounded and convex. In this setting, for $p, q \in (1, \infty)$ the solution and data spaces are given by

$$\begin{aligned} \mathbb{F} &:= L^p(0,T; (W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))') \times L^q(\Omega)), \\ \mathbb{E}_1 &:= L^p(0,T; W^{1,q}(\Omega) \cap L^q_{av}(\Omega)) \cap W^{1,p}(0,T; (W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'), \\ \mathbb{E}_2 &:= L^p(0,T; W^{2,q}_N(\Omega)) \cap W^{1,p}(0,T; L^q(\Omega)), \\ \mathbb{E} &:= \mathbb{E}_1 \times \mathbb{E}_2. \end{aligned}$$

As before, we rewrite the system as an abstract evolution equation. For the involved operators it is shown in Section 2.4.2 that they have maximal regularity for $1 < q \leq 2$. In order to show that the nonlinear terms satisfy some Lipschitz conditions, we apply embeddings due to Sobolev and the mixed derivative theorem, which gives us further conditions on p and q. Summarizing, we obtain Theorem 4.1.1 on existence and uniqueness of periodic solutions to (PKS).

> **Theorem.** *Let* $\frac{3}{2} < q \leq 2$ and $\frac{1}{p} + \frac{3}{2q} < 1$. *Let* $T > 0$ *and assume that* $f = (f_1, f_2)^T \in \mathbb{F}$ *is* T*-periodic.*
> *Then there is* $r_0 > 0$ *such that for any* $r \in (0, r_0)$ *there exists* $\delta = \delta(r) > 0$ *such that if* $\|f\|_{\mathbb{F}} < \delta$, *then there exists a* T*-periodic solution* $w = (n, c)^T \in \mathbb{E}$ *to* (PKS), *which is unique in* $\overline{B_{\mathbb{E}}}(0, r)$.

In Section 4.2 we consider the initial value problem (KS) and show local well-posedness by applying the abstract theory introduced in Subsection 2.5.2. Furthermore, the generalized principle of linearized stabilty yields the existence of global solutions for small data.

Bidomain equations

There is a long tradition of mathematical models describing the propagation of impulses in electrophysiology starting with the pioneering work from Hodgkin and Huxley in the 1950s. Introduced for the first time by Tung [78] in 1978, the *bidomain equations* became a well established system for describing electrophysiological wave propagation in the myocardium. This system is given by

$$\text{(BDE)} \qquad \begin{cases} \partial_t u - \operatorname{div}(\sigma_i \nabla u_i) + f(u, w) = I_i & \text{in } (0, \infty) \times \Omega, \\ \partial_t u + \operatorname{div}(\sigma_e \nabla u_e) + f(u, w) = -I_e & \text{in } (0, \infty) \times \Omega, \\ \partial_t w + g(u, w) = 0 & \text{in } (0, \infty) \times \Omega, \\ u = u_i - u_e & \text{in } (0, \infty) \times \Omega, \\ \sigma_i \nabla u_i \cdot \nu = 0, \, \sigma_e \nabla u_e \cdot \nu = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ u(0) = u_0, \, w(0) = w_0 & \text{in } \Omega. \end{cases}$$

In this system, $\Omega \subset \mathbb{R}^d$ is the physical domain representing the myocardium. The unknowns u_i and u_e model the intra- and extracellular electric potentials and their difference u indicates the transmembrane potential. The gating variable w corresponds to the ionic transport through the cell membrane. The conductivity matrices $\sigma_i(x)$ and $\sigma_e(x)$ model the anisotropic properties of the intra- and extracellular tissue parts, respectively whereas the intra- and extracellular stimulation currents are denoted by $I_i(t,x)$ and $I_e(t,x)$.

The nonlinear terms f and g describe the ionic transport. A prototype example is the model by FitzHugh–Nagumo [27] which is given by

$$\begin{aligned} f(u,w) &= u(u-a)(u-1)+w, \\ g(u,w) &= -\varepsilon(ku-w), \end{aligned}$$

with $0 < a < 1$ and k, $\varepsilon > 0$.

Even though the bidomain equations where already introduced in 1978, first analytic results where obtained only in 2002 by Colli Franzone and Savaré [19] who showed the existence and uniqueness of weak and strong solutions to the bidomain equations with ionic transport of FitzHugh–Nagumo type. Extensions of their results to more general ionic models were given by Veneroni [80].

In [13] a new perspective was given by Bourgault, Cordière, and Pierre in 2009 who transformed the bidomain system into an abstract evolution equation by introducing the so-called *bidomain operator*, a non-negative and selfadjoint operator, in the L^2-setting for the first time. Using this representation, they showed the existence and uniqueness of a local strong solution as well as the existence of a global weak solution. Their results contained a significant larger class of ionic models as before, including for example the models by Aliev–Panfilov [3] and Rogers–McCulloch [74]. Uniqueness and further regularity for the weak solutions was shown by Kunisch and Wagner [56] under some additional assumptions.

First results in the L^p-setting were given by Giga and Kajiwara [33]. They defined the bidomain operator in said setting and proved that the negative of the operator generates a bounded analytic semigroup in $L^q(\Omega)$ for $q \in (1,\infty]$. Then, they used this results tho show the existence and uniqueness of a local strong solution.

Hieber and Prüss showed that the bidomain operator has a bounded $\mathcal{H}^\infty$-calculus on $L^q_{av}(\Omega)$ [39] as well as the existence and uniqueness of a global strong solution for FitzHugh–Nagumo ionic transport [42] for $I_i = I_e = 0$. Furthermore, they considered stability of homogeneous equilibria.

Most of the mentioned results deal with the question of well-posedness of the bidomain equations. Results on the dynamics of the solution are rare. We refer to Mori and Matano [67] who studied the stability of front solutions of the bidomain system.

Overview

Since the bidomain equations models electrical activities in the heart, it is a natural question whether they admit time-periodic solutions. It is one main objective of this thesis provide an answer this question. This is done via three different approaches. All three of them use the abstract formulation of (BDE) involving the bidomain operator which reads as

$$\text{(PABDE)} \qquad \begin{cases} u' + Au + f(u,w) = I & \text{in } \mathbb{R} \times \Omega, \\ w' + g(u,w) = 0 & \text{in } \mathbb{R} \times \Omega, \\ u(t+T,x) = u(t,x) & \text{in } \mathbb{R} \times \Omega, \\ w(t+T,x) = w(t,x) & \text{in } \mathbb{R} \times \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ is supposed to be a bounded C^2-domain.

The two approaches in Section 5.2 yield the existence and *uniqueness* of strong periodic solutions for *small data*. The first strategy described in Subsections 5.2.1-5.2.3 relies on a periodic version of the classical Da Prato–Grisvard theorem. This classical theorem yields maximal L^p-regularity for parabolic evolution equations in the real interpolation space $D_{\mathcal{A}}(\theta,p)$ given by

$$D_{\mathcal{A}}(\theta,p) := \left\{ x \in X : [x]_{\theta,p} := \left(\int_0^\infty \|t^{1-\theta}\mathcal{A}\mathrm{e}^{-t\mathcal{A}}x\|_X^p \frac{\mathrm{d}t}{t} \right)^{1/p} < \infty \right\}$$

provided $-\mathcal{A}$ is the generator of a bounded analytic semigroup. The key result for this approach is the extension of the Da Prato–Grisvard theorem to the time-periodic problem

$$\text{(PACP)} \qquad \begin{cases} u'(t) + \mathcal{A}u(t) = f(t), & t \in \mathbb{R}, \\ u(t) = u(t+T), & t \in \mathbb{R} \end{cases}$$

for a given T-periodic function $f : \mathbb{R} \to D_{\mathcal{A}}(\theta,p)$. This is done in Theorem 5.2.3. We emphasize that the case $p = 1$ is included.

Then, we employ the contraction mapping principle to extend this result to a semilinear setting. Finally, we apply this *semilinear version of the periodic Da Prato–Grisvard theorem* for the solution and data spaces

$$
\begin{aligned}
\mathbb{E} =& \{u \in W^{1,p}(0,T; D_A(\theta,p)) : Au \in L^p(0,T; D_A(\theta,p)) \text{ and } u(0) = u(T)\} \\
& \times \{w \in W^{1,p}(0,T; D_A(\theta,p)) : w(0) = w(T)\}, \\
\mathbb{F} =& L^p(0,T; D_{\mathcal{A}}(\theta,p))
\end{aligned}
$$

to the bidomain equations subject to various models for the ionic transport, including, e.g., the protoype FitzHugh–Nagumo model. This yields the existence and uniqueness of strong time-periodic solutions provided the external applied currents are sufficiently small. For the bidomain system with FitzHugh–Nagumo transport this is done in Theorem 5.2.14.

Theorem. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded C^2-domain. Let $2\theta \in (0,1)$, $1 \leq p < \infty$, and $1 < q < \infty$ satisfy $2\theta > d/q$ or, if $p = 1$ let $2\theta \geq d/q$. Assume $I : \mathbb{R} \to D_A(\theta,p)$ is a T-periodic function satisfying $I_{|(0,T)} \in \mathbb{F}$ for some $\theta \in (0,1/2)$ and $T > 0$.*

a) Then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}} < C(R)$, the equation (PABDE) *admits a unique T-periodic strong solution (u,w) with $(u,w)_{|(0,T)} \in \overline{B_{\mathbb{E}}}((0,0),R)$.*

b) If condition ($\mathrm{S_{FN}}$) *is satisfied, then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}} < C(R)$, the equation* (PABDE) *admits a unique T-periodic strong solution (u,w) with $(u,w)_{|(0,T)} \in \overline{B_{\mathbb{E}}}((u_3,w_3),R)$.*

Note that the bidomain operator A a priori does not satisfy the assumption $0 \in \rho(A)$, but after a linearization around suitable stable stationary solutions it is shown that for the resulting 2×2 operator matrix this assumption is fulfilled. These results have been obtained from a joint work with Matthias Hieber, Naoto Kajiwara, and Patrick Tolksdorf. They were published in [38].

An comparable approach in Subsection 5.2.4 uses the *semilinear Arendt–Bu theorem*, which combines maximal periodic L^p-regularity with the contraction mapping principle, in place of the periodic Da Prato–Grisvard theorem. Hence, in this approach the underlying ground space is $L^q(\Omega)$

instead of the real interpolation space $D_{\mathcal{A}}(\theta, p)$. This theorem is applied to the linearized version of the bidomain equations to get the existence and uniqueness of strong-time periodic solutions for the same models as before. Again, we have to assume smallness for the external data.

The third approach in Section 5.3 deals with this issue and allows for external currents *without restriction of the size*, however we lose the uniqueness of the solution. In contrast to the other strategies, this time we work in the L^2-L^2-setting. First, in Theorem 5.3.3 we establish the existence of a weak time-periodic solution to the bidomain equations assuming the nonlinear functions f and g satisfy some growth conditions.

> **Theorem.** *Let $T > 0$. Then for every T-periodic function $I \in L^2(0, T; (H^1(\Omega))')$ there exists at least one weak T-periodic solution to* (PABDE)*.*

The existence of the weak periodic solution is shown via a Galerkin approximation combined with the fixed point theorem of Brouwer. In order to show the existence of a strong periodic solution to (BDE) subject to arbitrary large forces, we consider the *weak periodic* solution (v, z) as a weak solution to the *initial value* problem with initial data $v(t_0)$ and $z(t_0)$ for some $t_0 > 0$. The global well-posedness result for FitzHugh–Nagumo type transport by Colli Franzone and Savaré gives us the existence of a strong solution (u, w) to this initial value problem. Then, a weak-strong uniqueness argument shows that (v, z) and (u, w) coincide and therefore we obtain the existence of a strong time-periodic solution to (BDE) with FitzHugh–Nagumo type ionic transport which corresponds to Theorem 5.3.8.

> **Theorem.** *Let $d = 3$, $T > 0$, and $I_{i,e} \in L^2(0, T; L^2(\Omega))$ be T-periodic with $I_i + I_e \in W^{1,1}(0, T; L^2(\Omega))$ and $\int_\Omega (I_i + I_e)\, \mathrm{d}x = 0$ for a.e. t.*
> *Then there exists a strong time-periodic solution*
> $$(u, w) \in (W^{1,2}(0, T; H) \cap L^2(0, T; H^2(\Omega)) \cap L^4(Q)) \times C^1(0, T; H)$$
> *to the bidomain equations with FitzHugh–Nagumo type transport.*

The results from Section 5.3 have been obtained from a joint work with Yoshikazu Giga and Naoto Kajiwara and were published in [34].

Liquid crystals

Liquid crystals are a state of matter having properties between those of a usual liquid and a solid crystal, e.g., a liquid crystal may flow like a fluid whereas its molecules are orientated in the way of a crystal. A well-known application are *liquid crystal displays (LCD)*. One famous model for describing the flow of nematic liquid crystals is the *Ericksen-Leslie model* which arises from the pioneering work of Ericksen and Leslie in the 1960s [25, 61]. This model couples the evolution equation for the molecular orientation (the solid part) to the Navier–Stokes equations (the liquid part). However, in this model the molecular orientation is described by a vector $d \in \mathbb{R}^d$ of unit length which excludes the case of biaxial liquid crystals.

In order to cover also biaxial liquid crystal, Beris and Edwards [11] presented a model in which the unit vector d is replaced by the so-called *Q-tensor*, a symmetric, traceless $d \times d$-matrix Q, i.e.,

$$Q(x) \in \mathbb{S}^d_{0,\mathbb{R}} := \{Q \in \mathbb{R}^{d\times d} : Q = Q^T, \operatorname{tr} Q = 0\}.$$

For a bounded domain $\Omega \subset \mathbb{R}^d$ the *Beris–Edwards model* is given by

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = \operatorname{div}(\tau(Q) + \sigma(Q)) & \text{in } (0,\infty)\times\Omega, \\ \operatorname{div} u = 0 & \text{in } (0,\infty)\times\Omega, \\ \partial_t Q + (u\cdot\nabla)Q - S(\nabla u, Q) = \Gamma H(Q) & \text{in } (0,\infty)\times\Omega, \\ (u, \partial_{\vec{\nu}} Q) = (0,0) & \text{on } (0,\infty)\times\partial\Omega, \\ (u(0), Q(0)) = (u_0, Q_0) & \text{in } \Omega, \end{cases}$$

The unknowns of this system are the velocity $u\colon (0,\infty)\times\Omega \to \mathbb{R}^d$, the pressure $p\colon (0,\infty)\times\Omega \to \mathbb{R}$, and the Q-tensor $Q\colon (0,\infty)\times\Omega \to \mathbb{S}^d_{0,\mathbb{R}}$ which describes the molecular orientation. The symmetric and antisymmetric part of the stress tensor are described by $\tau = \tau(Q,\xi)$ and $\sigma = \sigma(Q)$, respectively. Furthermore, $S = S(\nabla u, Q, \xi)$ describes how the flow gradient rotates and stretches the molecular orientation and the parameter $\xi \in \mathbb{R}$ describes the ratio of tumbling and aligning effects.

Compared to the Ericksen–Leslie model, there are much less results available for the Beris–Edwards model. Furthermore, those results heavily depend on the space dimension and the parameter ξ.

First results were obtained by Paicu and Zarnescu. For the whole space case and the choice $\xi = 0$ they showed in [69] the existence of a global weak solution to the Beris–Edwards model in two and three space dimensions and extended their results for small $\xi \neq 0$ in [68]. Wilkinson [81] showed the existence of weak solutions for general $\xi \in \mathbb{R}$.

For the case $\xi = 0$ the existence and uniqueness of local strong solutions was shown by Abels, Dolzmann, and Liu for different boundary conditions [1]. For general $\xi \in \mathbb{R}$ global well-posedness was shown in [15] in two space dimensions. Very recently, Wrona [84] obtained for the first time results for general ξ in three dimensions. He showed the existence and uniqueness of a local strong solution as well as global existence for small data.

Overview

Relying on the results given by Wrona [84] it is the aim to show the existence and uniqueness of strong time-periodic solutions to the time-periodic Beris–Edwards model

(BE)

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = \operatorname{div}(\tau(Q) + \sigma(Q)) + g_1(t) & \text{in } \mathbb{R} \times \Omega, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R} \times \Omega, \\ \partial_t Q + (u \cdot \nabla) Q - S(\nabla u, Q) = \Gamma H(Q) + g_2(t) & \text{in } \mathbb{R} \times \Omega, \\ (u, \partial_{\vec{\nu}} Q) = (0, 0) & \text{on } \mathbb{R} \times \partial\Omega, \\ (u(0), Q(0)) = (u(T), Q(T)) & \text{in } \Omega, \end{cases}$$

provided the system is innervated by T-periodic external forces $g_1(t)$ and $g_2(t)$. Here the domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a bounded domain with C^3-boundary.

Therefore, (BE) is rewritten as a quasilinear evolution equation. The involved operator matrix is known to be invertible and to admit the property of maximal regularity under certain conditions. Combining this with Lipschitz estimates for the nonlinear terms, we obtain the existence and uniqueness of a strong periodic solution for general $\xi \in \mathbb{R}$ and sufficiently small g_1 and g_2. This is done in Theorem 6.2.1.

In Section 6.3, we consider modified versions of (BE) and show that for these modified version we obtain again the existence and uniqueness of strong time-periodic solutions.

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Thank you!

Zusammenfassung in deutscher Sprache

Das Hauptanliegen dieser Dissertation ist die Erforschung von verschiedenen Modellen, welche ihren Ursprung in der mathematischen Biologie und Fluidmechanik haben, im zeitperiodischen Setting. Wir untersuchen sowohl das *klassische Keller–Segel Modell* für Chemotaxis als auch dessen Kopplung zu den Navier–Stokes Gleichungen, welche die Strömung von viskosen Fluiden beschreiben. Das zweite betrachtete Modell, das *Bidomain System*, beschreibt die Ausbreitung von elektrophysiologischen Wellen im Herzen. Als letztes Modell untersuchen wir das *Beris–Edwards Modell* für nematische Flüssigkristalle.

Chemotaxis

Wir betrachten in dieser Arbeit ein Chemotaxis-Fluid Modell in einem beschränkten Gebiet $\Omega \in \mathbb{R}^d$ mit glattem Rand. Hauptresultat der Untersuchung dieses Modells sind die *Existenz und Eindeutigkeit von starken, zeitperiodischen Lösungen* unter der Voraussetzung, dass die externen Kräfte hinreichend klein sind. Wir schreiben das Modell als abstrakte Evolutionsgleichung und verwenden das quasilineare Arendt–Bu Theorem. Hierzu zeigen wir, dass der zugehörige lineare Operator die Eigenschaft der maximalen periodischen L^p-Regularität besitzt und dass die nichtlinearen Terme gewisse Lipschitz-Abschätzungen erfüllen. Des Weiteren zeigen wir die lokale Wohlgestelltheit des zugehörigen Anfangswertproblems.

Außerdem untersuchen wir mit einem ähnlichen Ansatz das *klassische*

Keller–Segel Modell auf einem beschränkten und konvexen Gebiet $\Omega \subset \mathbb{R}^3$. Das Hauptresultat ist hier ebenfalls die Existenz und Eindeutigkeit von starken, zeitperiodischen Lösungen. Für das zugehörige Anfangswertproblem zeigen wir Eindeutigkeit und Existenz von lokalen Lösungen sowie die globale Existenz für kleine Anfangsdaten.

Bidomain-Gleichungen

Die Bidomain-Gleichungen im periodischen Setting werden aus zwei verschiedenen Gesichtspunkten betrachtet.

Zum einen zeigen wir *Existenz und Eindeutigkeit* von starken, zeitperiodischen Lösungen für *kleine externe Kräfte*. Hierzu beweisen wir zunächst eine periodische Version des klassischen Theorems von Da Prato und Grisvard, welches uns die Lösung einer abstrakten linearen Evolutionsgleichung in reellen Interpolationsräumen liefert, vorausgesetzt der lineare Operator ist invertierbar und der Generator einer beschränkten analytischen Halbgruppe. Mit Hilfe des Banachschen Fixpunktsatzes erweitern wir dieses Resultat auf semilineare Gleichungen und erhalten so im Anschluss das gewünschte Resultat für die Bidomain-Gleichungen. Ein vergleichbarer Ansatz beruht auf dem semilinearen Arendt–Bu Theorem und liefert periodische Lösungen im klassichen $L^p - L^q$ Setting der maximalen Regularität.

Zum anderen zeigen wir Existenz von starken, periodischen Lösungen *ohne Größenbeschränkungen* an die externen Kräfte. Als Preis verlieren wir bei diesem Ansatz die Eindeutigkeit. Wir zeigen zunächst die Existenz von schwachen, periodischen Lösungen mit Hilfe der Galerkin-Approximation und dem Fixpunktsatz von Brouwer. Unter Verwendung der globalen Wohlgestelltheit des Anfangswertproblems und eines schwach-stark Eindeutigkeitsarguments erhalten wir schließlich die Existenz von starken periodischen Lösungen für die Bidomain-Gleichungen für große Daten.

Flüssigkristalle

Wir betrachten das Beris–Edwards Q-Tensor Modell für nematische Flüssigkristalle mit beliebigem Parameter $\xi \in \mathbb{R}$. Um die aktuellen Resultate bezüglich maximaler Regularität nutzen zu können, schreiben wir das Modell als quasilineare Evolutionsgleichung und kombinieren die maximale Regularität mit dem quasilinearen Arendt–Bu Theorem. Dies liefert uns die Existenz und Eindeutigkeit von starken, periodischen Lösungen für kleine externe Kräfte.

Preliminaries

CHAPTER 1

Preliminaries

In this chapter we collect some notations and concepts which are used througout this thesis.

1.1 Basic Notation

Sets of numbers

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural numbers, integers, real and complex numbers, respectively. Furthermore, we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a complex number $z \in \mathbb{C}$ the complex conjugate is denoted by $\overline{z}$.

Multi-indices

Let $n \in \mathbb{N}$. For multi-indices $\alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{N}_0^n$ and $\beta = (\beta_1, \ldots \beta_n) \in \mathbb{N}_0^n$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and by $\alpha \leq \beta$ we mean $\alpha_j \leq \beta_j$ for all $j = 1, \ldots, n$.

Linear algebra

The scalar product in $\mathbb{R}^d$, $d \geq 2$, is denoted by the $\cdot$ symbol. We write $\mathcal{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ for the unit sphere in $\mathbb{R}^d$. Let $\mathbb{K}$ be a field, then the trace and the transpose of a matrix $A \in \mathbb{K}^{d\times d}$ are denoted by $\operatorname{tr}(A)$ and A^T, respectively. While $\mathbb{M}^d_{0,\mathbb{K}}$ stands for the set of traceless $d \times d$-matrices, the set $\mathbb{S}^d_{0,\mathbb{K}} \subset \mathbb{M}^d_{0,\mathbb{K}}$ is the restriction to those matrices which are symmetric. Both spaces are closed subspaces of the Hilbert space $\mathbb{K}^{d\times d}$ with the usual scalar product $\langle A, B\rangle_{\mathbb{K}^{d\times d}} = \operatorname{tr}(A\overline{B^T})$. For a vector $v \in \mathbb{C}^d$ we write $v \otimes v$ for the symmetric matrix given by $(v \otimes v)_{i,j} = v_i v_j$, for $i, j = 1, \ldots, d$.

Vector spaces

In a normed vector space X the norm is denoted by $\|\cdot\|_X$. Note that sometimes the subscript is dropped or modified if there is no danger of confusion. Let X, Y be normed spaces, then we denote by $\mathcal{L}(X, Y)$ the space of bounded linear mappings from X to Y, and if $X = Y$, we simply write $\mathcal{L}(X) = \mathcal{L}(X, X)$. The topological dual space of X is denoted by $X' = \mathcal{L}(X, \mathbb{K})$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is the underlying field. By ${}_{X'}\langle\cdot,\cdot\rangle_X$ we denote the dual pairing between X and X'.

For given $R > 0$ and $x \in X$ we denote by $B_X(x, R) := \{y \in X : \|x-y\| < R\}$ the open ball with center x and radius R. The corresponding closed ball is denoted by $\overline{B_X}(x, R)$. If $X = \mathbb{R}^d$ we may drop the subscript X.

Time-periodicity

Let $0 < T < \infty$ and X be a Banach space. We call a measurable function $f : \mathbb{R} \to X$ *(time-)periodic of period* T or T*-(time-)periodic* if $f(t) = f(t + T)$ holds true for almost all $t \in \mathbb{R}$.

If we write $f \in L^p(0, T; X)$ is T-periodic, we implicitly mean that f is a T-periodic function $f : \mathbb{R} \to X$ which satisfies $f_{|(0,T)} \in L^p(0, T; X)$.

Domains

A set $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ is called domain if it is open and connected. If it is furthermore contained in a ball $B(0, R)$ for some radius $R > 0$ we say that Ω is bounded.

If the boundary $\partial\Omega$ can locally be represented as the graph of a Lipschitz continuous function, we say that Ω is a Lipschitz domain or that Ω has Lipschitz boundary. For $k \in \mathbb{N}$ we say that Ω is a domain of class C^k, with C^k-boundary, or a C^k-domain, if it can be locally represented by the graph of a function which is k-times continuously differentiable.

For a given domain $\Omega \subset \mathbb{R}^d$ and time $T > 0$ we set $Q = \Omega \times (0, T)$.

Linear operators

Let A be a linear operator in X. Then, $D(A)$ denotes its domain, $\sigma(A)$ its spectrum, and $\rho(A)$ its resolvent set. The operator is said to be closed if its graph $G(A) = \{(x, Ax) \in X^2 : x \in D(A)\}$ is a closed subset of X^2. The range and the kernel of A are denoted by $R(A)$ and $N(A)$, respectively.

1.2 Function Spaces

Continuous functions

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ be an open set, X be a normed vector space and $k \in \mathbb{N}_0 \cup \{\infty\}$. Then, $C^k(\Omega; X)$ denotes the set of all continuous functions $f : \Omega \to X$ whose partial derivatives $\partial^\alpha f$ are continuous in Ω for each multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$.

Moreover, the space $C^k(\overline{\Omega}; X)$ consists of all functions $f \in C^k(\Omega; X)$, such that the partial derivative $\partial^\alpha f$ has a continuous extension to $\overline{\Omega}$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$. Furthermore, $BUC(\Omega; X)$ denotes the space of all bounded uniformly continuous functions from Ω to X.

If $X = \mathbb{R}$ we simply write, e.g., $C^k(\Omega)$ instead of $C^k(\Omega; \mathbb{R})$. By $C_c^\infty(\Omega)$ we denote the set of all compactly supported functions in $C^\infty(\Omega)$.

Bochner–Lebesgue spaces

Let $\Omega \subset \mathbb{R}^d$ be an open set and X be a Banach space. Then, for $p \in [1, \infty]$, the *Bochner–Lebesgue space* $L^p(\Omega; X)$ denotes the space of all equivalence classes of Bochner–Lebesgue measurable functions $f : \Omega \to X$ such that $\|f\|_{L^p(\Omega;X)} < \infty$, where

$$\begin{aligned} \|f\|_{L^p(\Omega;X)} &:= \left(\int_\Omega \|f(x)\|_X^p \mathrm{d}x\right)^{1/p}, && \text{if } p < \infty, \\ \|f\|_{L^\infty(\Omega;X)} &:= \operatorname*{ess\,sup}_{x\in\Omega} \|f(x)\|_X, && \text{if } p = \infty. \end{aligned}$$

For details concerning the Bochner integral, we refer, e.g., to [86, Section V.5]. If $\Omega = (a,b) \subset \mathbb{R}$ for some a, $b \in \mathbb{R}$, we write $L^p(a,b;X)$ instead of $L^p(\Omega;X)$. Morevover, if $X = \mathbb{K}^d$ for $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, we usually write $L^p(\Omega)^d$ instead of $L^p(\Omega;\mathbb{K}^d)$ and drop the superscript if it is clear from the context.

We write $|\Omega|$ for the Lebesgue measure of the set Ω. We denote by $L^p_{loc}(\Omega)$ the set of all measurable functions $f : \Omega \to \mathbb{R}$ which are in $L^p(K)$ for each compact $K \subset \Omega$ and by $L^p_{av}(\Omega) := \{u \in L^p(\Omega) : \int_\Omega u \,\mathrm{d}x = 0\}$ the set of functions in $L^p(\Omega)$ with mean zero.

Sobolev spaces

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ be an open set, X a Banach space, $k \in \mathbb{N}_0$, $p \in [1,\infty]$, and $\alpha \in \mathbb{N}_0^d$ a multi-index. Then, the *Sobolev space* $W^{k,p}(\Omega;X)$ denotes the space of all equivalence classes $f : \Omega \to X$ with weak derivatives $\partial^\alpha f \in L^p(\Omega;X)$ of order $|\alpha| \le k$, which is equipped with the norm

$$\begin{aligned} \|f\|_{W^{k,p}(\Omega;X)} &:= \left(\Sigma_{|\alpha|\le k} \|\partial^\alpha f\|^p_{\mathrm{L}^p(\Omega;X)}\right)^{1/p}, && \text{if } p < \infty, \\ \|f\|_{W^{k,\infty}(\Omega;X)} &:= \max_{|\alpha|\le k} \|\partial^\alpha f\|_{L^\infty(\Omega;X)}, && \text{if } p = \infty. \end{aligned}$$

As above, if $X = \mathbb{R}^d$ we simply write $W^{k,p}(\Omega)^d$ or just $W^{k,p}(\Omega)$.

Schwartz space and tempered distributions

Let X be a Banach space. Then, the space of X-valued *Schwartz functions* $\mathcal{S}(\mathbb{R}^d;X)$ is defined via the usual family of seminorms. Moreover, the space of X-valued *tempered distributions* then is given by $\mathcal{S}'(\mathbb{R}^d;X) := \mathcal{L}(\mathcal{S}(\mathbb{R}^d;\mathbb{C});X)$. For a detailed introduction see, e.g., [4, Section III.4.1.2].

Fourier transform

The *Fourier transform* $\mathcal{F}$ on $\mathcal{S}(\mathbb{R}^d;\mathbb{C})$ is defined via

$$(\mathcal{F}f)(\xi) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\mathrm{d}x, \quad f\in\mathcal{S}(\mathbb{R}^d;\mathbb{C}),\ \xi\in\mathbb{R}^d.$$

For a Banach space X, the *Fourier transform* $\mathcal{F}$ on $\mathcal{S}'(\mathbb{R}^d;X)$ is given by

$$(\mathcal{F}\varphi)f := \varphi(\mathcal{F}f), \quad \varphi\in\mathcal{S}'(\mathbb{R}^d,X),\ f\in\mathcal{S}(\mathbb{R}^d;\mathbb{C}).$$

Bessel potential spaces

Let $q\in[1,\infty)$, $s\in\mathbb{R}$, and $m_s(x)=(1+|x|^2)^{s/2}$. Then the *X-valued Bessel potential space* $H^{s,q}(\mathbb{R}^d;X)$ is defined as

$$H^{s,q}(\mathbb{R}^d;X) := \{f\in\mathcal{S}'(\mathbb{R}^d;X) : \mathcal{F}^{-1}(m_s\mathcal{F}f)\in L^q(\mathbb{R}^d;X)\}$$

and it is endowed with the norm

$$\|f\|_{H^{s,q}(\mathbb{R}^d;X)} := \|\mathcal{F}^{-1}(m_s\mathcal{F})f\|_{L^q(\mathbb{R}^d;X)}.$$

If $q=2$ we omit the q and just write $H^s(\mathbb{R}^d;X)$ instead of $H^{s,2}(\mathbb{R}^d;X)$. Let $\Omega\subset\mathbb{R}^d$, $d\in\mathbb{N}$, be a domain. Then, the corresponding space is given by

$$H^{s,q}(\Omega;X) := \{f|_\Omega : f\in H^{s,q}(\mathbb{R}^d;X)\},$$

equipped with the norm

$$\|f\|_{H^{s,q}(\Omega;X)} := \inf\{\|g\|_{H^{s,q}(\mathbb{R}^d,X)} : g\in H^{s,q}(\mathbb{R}^d;X), f=g|_\Omega\}.$$

As above, if $X=\mathbb{R}^d$ we simply write $H^{k,p}(\Omega)^d$ or just $H^{k,p}(\Omega)$.

Remark 1.2.1. a) Let $k\in\mathbb{N}$. Then the space $H^{k,q}(\mathbb{R}^n)$ coincides with the space $W^{k,q}(\mathbb{R}^n)$.

b) The same holds true for domains with sufficiently regular boundary, e.g., with Lipschitz boundary. For these domains, there exists an extension operator from $W^{k,p}(\Omega)$ into $W^{k,q}(\mathbb{R}^d)$ (cf. e.g. [75, Chapter VI, Theorem 5]). Hence, the space $H^{k,q}(\Omega)$ coincides with the space $W^{k,q}(\Omega)$ for $k\in\mathbb{N}$.

Besov spaces

Let $\varphi = (\varphi_n(x))_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ be a sequence of Schwartz functions satisfying

- $\operatorname{supp}\varphi_0 \subset \overline{B}(0,2)$,
- $\operatorname{supp}\varphi_n \subset \{x \in \mathbb{R}^d : 2^{n-1} \leq x \leq 2^{n+1}\}, \quad n \geq 1,$
- for every multi-index $\alpha \in \mathbb{N}^d$ there exists a constant $C_\alpha > 0$ such that
$$2^{|\alpha|n}\|\partial^\alpha\varphi_n(x)\|_{L^\infty(\mathbb{R}^d)} \leq C_\alpha \quad \text{for all } n \in \mathbb{N},\ x \in \mathbb{R}^d,$$
- for every $x \in \mathbb{R}^d$ holds $\sum_{n=0}^\infty \varphi_n(x) = 1$.

Then, for $p, q \in [1,\infty]$ and $s \in \mathbb{R}$ the *X-valued Besov space* $B^s_{q,p}(\mathbb{R}^d; X)$ is defined as

$$B^s_{q,p}(\mathbb{R}^d; X) := \{f \in \mathcal{S}'(\mathbb{R}^d, X) : \left(2^{sn}\|\mathcal{F}^{-1}\varphi_n\mathcal{F}f\|_{L^q(\mathbb{R}^d;X)}\right)_{n\in\mathbb{N}} \in l^q(\mathbb{N})\}$$

and it is endowed with the following norm

$$\|f\|_{B^s_{q,p}(\mathbb{R}^d;X)} := \|(2^{sn}\|\mathcal{F}^{-1}\varphi_n\mathcal{F}f\|_{L^q(\mathbb{R}^d;X)})_{n\in\mathbb{N}}\|_{l^q(\mathbb{N})}.$$

For $\Omega \subset \mathbb{R}^d$ the corresponding space $B^s_{q,p}(\Omega; X)$ is defined in the same way as for the Bessel potential space. For convenience, we drop the comma between q and p if there is no danger of confusion.

Boundary conditions

For the function spaces introduced above, we define in the usual way their subspaces satisfying Dirichlet or Neumann boundary conditions, see, e.g., [5]. To be more precise, in the case of Neumann boundary conditions, for $k \in \{2,3\}$ we define

$$W^{k,q}_N(\Omega) := \{g \in W^{k,q}(\Omega) : \frac{\partial g}{\partial \nu} = 0 \text{ on } \partial\Omega\}$$

and for $s > 1 + 1/q$ we set

$$B^s_{qp,N}(\Omega) := \{g \in B^s_{qp}(\Omega) : \frac{\partial g}{\partial \nu} = 0 \text{ on } \partial\Omega\}.$$

For homogeneous Dirichlet boundary conditions we set

$$W^{1,q}_0(\Omega) = \{g \in W^{1,q}(\Omega) : g = 0 \text{ on } \partial\Omega\}$$

and for $s > 1/q$ we define

$$B^s_{qp,0}(\Omega) := \{g \in B^s_{qp}(\Omega) : g = 0 \text{ on } \partial\Omega\}.$$

1.3 Interpolation of Function Spaces

In this section we briefly introduce interpolation spaces and characterize Besov spaces via real interpolation. For a detailed introduction, we refer, e.g., to the books of Adams and Fournier [2], Bergh and Löfström [10], Lunardi [65], and Triebel [77].

In the following, let X_0 and X_1 be Banach spaces. We call (X_0, X_1) an *interpolation couple* if X_0 and X_1 are embedded into a common topological Hausdorff vector space. For an interpolation couple (X_0, X_1) we can consider their sum $X_0 + X_1$ endowed with the norm

$$\|x\|_{X_0+X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x + x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$$

and their intersection $X_0 \cap X_1$ endowed with the norm

$$\|x\|_{X_0 \cap X_1} = \|x_0\|_{X_0} + \|x_1\|_{X_1}.$$

Then, a space X is called intermediate space of the interpolation couple (X_0, X_1) if $X_0 \cap X_1 \subset X \subset X_0 + X_1$ together with continuous inclusions. An intermediate space X is called *interpolation space* if for every $T \in \mathcal{L}(X_0+X_1)$ whose restriction to X_0 belongs to $\mathcal{L}(X_0)$ and whose restriction to X_1 belongs to $\mathcal{L}(X_1)$, we have that the restriction of T to X belongs to $\mathcal{L}(X)$.

Complex interpolation

Let (X_0, X_1) be an interpolation couple of complex Banach spaces. Define the strip

$$S := \{z \in \mathbb{C} : \mathrm{Re}(z) \in [0,1]\}.$$

The set $\mathcal{F}(X_0, X_1)$ is the set of all functions $f : S \to X_0 + X_1$ which satisfy

- f is continuous on S and analytic on the interior of S,
- the functions $\mathbb{R} \to X_0$, $t \mapsto f(it)$ and $\mathbb{R} \to X_1$, $t \mapsto f(1+it)$ are continuous and

$$\|f\|_{\mathcal{F}(X_0,X_1)} := \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_1}\} < \infty.$$

Definition 1.3.1. Let $\theta \in [0,1]$ and (X_0, X_1) be an interpolation couple of complex Banach spaces. The *complex interpolation space between X_0 and X_1 with parameter θ* is the set

$$[X_0, X_1]_\theta := \{f(\theta) : f \in \mathcal{F}(X_0, X_1)\}$$

equipped with the norm

$$\|g\|_{[X_0,X_1]_\theta} := \inf\{\|f\|_{\mathcal{F}(X_0,X_1)} : f \in \mathcal{F}(X_0, X_1) \text{ with } f(\theta) = g\}.$$

An important application of this concept is the complex interpolation of Sobolev spaces. Let s_0, $s_1 \in \mathbb{R}$ with $s_0 \neq s_1$, $q_0, q_1 \in [1, \infty]$, and $\theta \in (0,1)$. Then, for $s = (1-\theta)s_0 + \theta s_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$ we obtain

$$[H^{s_0,q_0}(\mathbb{R}^d), H^{s_1,q_1}(\mathbb{R}^d)]_\theta = H^s_q(\mathbb{R}^d).$$

For more details we refer to [10, Chapter 5 and 6]. For applications to domains with different types of boundary conditions, see, e.g., [5] and [72].

Real interpolation

Let (X_0, X_1) be an interpolation couple. Then, for all $t > 0$ and $x \in X_0 + X_1$ the K*-functional* is defined by

$$K(t, x, X_0, X_1) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\}.$$

Definition 1.3.2. Let $\theta \in (0, 1)$, $p \geq 1$, (X_0, X_1) be an interpolation couple. Then the *real interpolation space between X_0 and X_1 with parameter θ and p* is the set

$$(X_0, X_1)_{\theta,p} := \{x \in X_0 + X_1 : \|x\|_{(X_0,X_1)_{\theta,p}} < \infty\}$$

equipped with the norm

$$\|x\|_{(X_0,X_1)_{\theta,p}} := \left(\int_0^\infty \left(t^{-\theta} K(t, x, X_0, X_1)\right)^p \frac{\mathrm{d}t}{t}\right)^{1/p}.$$

As above, an important application is the real interpolation of Sobolev spaces. Let s_0, $s_1 \in \mathbb{R}$ with $s_0 \neq s_1$, $p, q \in [1, \infty]$, and $\theta \in (0, 1)$. Then, for $s = (1 - \theta)s_0 + \theta s_1$ we obtain

$$(H^{s_0,q}(\mathbb{R}^d), H^{s_1,q}(\mathbb{R}^d))_{\theta,p} = B^s_{q,p}(\mathbb{R}^d).$$

For more details we refer again to [10, Chapter 5 and 6]. For applications to domains with different types of boundary conditions, see, e.g., [5] and [72].

CHAPTER 2

Operator Theory

In this chapter, we provide a brief introduction to the theory of analytic semigroups, maximal regularity, and the $\mathcal{H}^\infty$-calculus. We give some examples of well-known operators satisfying these properties. These are used to prove embeddings which follow from the mixed derivative theorem. Finally, we recall results for the existence and uniqueness of solutions to abstract Cauchy problems as well as abstract time-periodic problems.

These results will be applied in the following chapters to various equations involving the operators mentioned before.

2.1 Analytic Semigroups

We recall two important results for bounded analytic semigroups. For a detailed introduction to the theory if semigroups, we refer, e.g., to [6, Chapter 3].

Let X be a Banach space. For $\theta \in (0, \pi)$ define the sector $\Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| < \theta\}$. Then, we have the following two results.

Proposition 2.1.1 ([6, Theorem 3.7.11])**.** *Let A be an operator on X. Then, A is the generator of a bounded analytic semigroup of angle $\theta \in (0, \pi/2]$ if and only if $\Sigma_{\theta+\pi/2} \subset \rho(A)$ and*

$$\sup_{\lambda \in \Sigma_{\theta+\frac{\pi}{2}-\varepsilon}} \|\lambda(\lambda - A)^{-1}\|_{\mathcal{L}(X)} < \infty \qquad \text{for all } \varepsilon > 0.$$

Proposition 2.1.2 ([6, Theorem 3.7.19])**.** *Let A be the generator of a bounded analytic semigroup T. Then, $T(t)x \in D(A)$ for all $t > 0$, $x \in X$ and*

$$\sup_{t>0} \|tAT(t)\|_{\mathcal{L}(X)} < \infty.$$

Note that the bounded analytic semigroup generated by A is a C_0-semigroup if and only if $D(A)$ is dense in X. The semigroup corresponding to A is usually denoted by $(\mathrm{e}^{tA})_{t\geq 0}$.

2.2 Maximal Regularity

In this section, we introduce the notion of maximal regularity for the Cauchy problem as well as maximal periodic regularity for the time-periodic problem. For more details we refer, e.g., to [71] for the Cauchy problem and to [7] for the time-periodic problem.

To this end, let X_0 and X_1 be Banach spaces with X_1 densely embedded in X_0 and $A : X_1 \to X_0$ be a closed linear operator. Let $p \in (1, \infty)$,

$T \in (0, \infty]$, and $f : (0,T) \to X_0$. We consider the Cauchy problem

$$\begin{cases} u'(t) - Au(t) = f, & t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{2.1}$$

We say that A has the property of *maximal (L^p-)regularity* if for any given $f \in L^p(0,T;X_0)$ there exists a unique $u \in H^{1,p}(0,T;X_0) \cap L^p(0,T;X_1)$ satisfying (2.1) almost everywhere in $(0,T)$ with $u_0 = 0$. Note that in the literature the Cauchy problem (2.1) is sometimes also written using $+Au(t)$ instead of $-Au(t)$ leading to a slightly different notion, i.e., it depends on the notion whether A or $-A$ enjoy the maximal regularity property.

If we consider (2.1) with an initial value $u_0 \in (X_0, X_1)_{1-1/p,p}$, we obtain the following estimate

$$\|u\|_{H^{1,p}(0,T;X_0)} + \|Au\|_{L^p(0,T;X_0)} \le C(\|u_0\|_{(X_0,X_1))_{1-1/p,p}} + \|f\|_{\mathrm{L}^p(0,T;X_0)}).$$

Next, we consider the following time-periodic problem

$$\begin{cases} u'(t) - Au(t) = f(t), & t \in (0,2\pi), \\ u(0) = u(2\pi). \end{cases} \tag{2.2}$$

We say that A admits *maximal periodic (L^p-)regularity* if for each $f \in L^p(0,2\pi;X_0)$ there exists a unique $u \in H^{1,p}(0,2\pi;X_0) \cap L^p(0,2\pi;X_1)$ satisfying (2.2) almost everywhere in $(0,2\pi)$. Note that it is no real restriction to consider the time period $(0,2\pi)$ since any time interval $(0,T)$ for any $T > 0$ can be reduced to that case by some scaling in time.

Due to the closed graph theorem, there exists a constant $C > 0$ such that

$$\|u\|_{H^{1,p}(0,2\pi;X_0)} + \|Au\|_{L^p(0,2\pi;X_0)} \le C\|f\|_{\mathrm{L}^p(0,2\pi;X_0)}.$$

Arendt and Bu characterized the relationship between maximal periodic L^p-regularity and maximal L^p-regularity for the Cauchy problem as follows.

Proposition 2.2.1 ([7, Theorem 5.1])**.** *Let X be a Banach space and $A : D(A) \to X$ be the generator of a C_0-semigroup on X. Then A admits maximal periodic L^p-regularity if and only if $1 \in \rho(\mathrm{e}^{2\pi A})$ and A admits maximal L^p-regularity to the corresponding Cauchy problem.*

2.3 $\mathcal{H}^\infty$-calculus

We introduce the bounded $\mathcal{H}^\infty$-calculus and in order to do so, the notion of sectorial operators. For details, we refer, e.g., to the books of Haase [35], Denk, Hieber, and Prüss [22], or the recent lecture notes [36].

Let A be a closed linear operator on a Banach space X and assume that $R(A)$ and $D(A)$ are dense in X. Then, A is called a *sectorial operator of angle* $\phi \in (0,\pi)$ if $\sigma(A) \subset \overline{\Sigma_\phi}$ and if for every $\phi' \in (\phi,\pi)$ there exists a constant $C > 0$ such that

$$\|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq C$$

for all $\lambda \in \mathbb{C} \setminus \overline{\Sigma_{\phi'}}$. The infimum of all $\phi \in (0,\pi)$ for which this holds true is called *spectral angle of* A and is denoted by ϕ_A.

In the following, let A be sectorial operator with spectral angle ϕ_A. For $\phi \in (0,\pi)$ we define the set of bounded holomorphic functions

$$\mathcal{H}^\infty(\Sigma_\phi) := \{f\colon \Sigma_\phi \to \mathbb{C} : f \text{ is bounded and holomorphic}\}$$

endowed with the norm

$$\|f\|_\infty^\phi = \sup\{|f(\lambda| : |\arg\lambda| < \phi\}.$$

Next, we consider functions $f \in \mathcal{H}^\infty(\Sigma_\phi)$ which have some decay at 0 and infinity. To be more precise, we assume that

$$|f(\lambda)| \leq C\frac{|\lambda|^\varepsilon}{|1+\lambda|^{2\varepsilon}}, \quad \lambda \in \Sigma_\phi \tag{2.3}$$

for some C, $\varepsilon > 0$ and define

$$\mathcal{H}_0^\infty(\Sigma_\phi) := \{f \in \mathcal{H}^\infty(\Sigma_\phi) : \text{there exists } C, \varepsilon > 0 \text{ such that (2.3) holds}\}.$$

Let $\phi \in (\phi_A,\pi)$ and $f \in \mathcal{H}_0^\infty(\Sigma_\phi)$. Then the mapping

$$f \mapsto f(A) := \frac{1}{2\pi i}\int_\gamma f(\lambda)(\lambda + A)^{-1}\mathrm{d}\lambda,$$

is called functional calculus (of A). Here γ denotes a path which rounds $\partial\Sigma_\psi$ counterclockwise for some $\psi \in (\phi_A, \phi)$.

Then, a sectorial operator A with spectral angle ϕ_A is said to have a *bounded* $\mathcal{H}^\infty(\Sigma_\phi)$*-calculus* for $\phi \in (\phi_A, \pi)$ if there exists a $C > 0$ such that

$$\|f(A)\|_{\mathcal{L}(X)} \leq C\|f\|_{\mathcal{H}^\infty(\Sigma_\phi)} \quad \text{for all } f \in \mathcal{H}_0^\infty(\Sigma_\phi).$$

As above, the infimum of all such ϕ is called the $\mathcal{H}^\infty$*-angle of* A and is denoted by ϕ_A^∞. The set of sectorial operators admitting a bounded $\mathcal{H}^\infty$-calculus on X is denoted by $\mathcal{H}^\infty(X)$.

The $\mathcal{H}^\infty$-calculus characterizes the fractional power spaces

$$X_\alpha = (D(A^\alpha), \|\cdot\|_\alpha), \quad \|\cdot\|_\alpha = \|x\| + \|A^\alpha x\|, \quad 0 < \alpha < 1$$

of a sectorial operator A on a Banach space X, see, e.g., [36, Theorem 1.5.6].

Proposition 2.3.1. *Let X be a Banach space and $A \in \mathcal{H}^\infty(X)$. Then*

$$X_\alpha \simeq [X, D(A)]_\alpha, \quad 0 < \alpha < 1,$$

where $[X, D(A)]_\alpha$ denotes the complex interpolation space of order α.

2.4 Examples

In this section, we consider different operators satisfying the properties described in the previous sections. Namely, we consider the Laplacian on different spaces and domains as well as the Stokes operator.

2.4.1 The Neumann-Laplacian on Smooth Domains

We begin with collecting some results for the Laplacian with Neumann boundary conditions on bounded smooth domains.

We start with the classical Neumann-Laplacian on $L^q(\Omega)$ and the first result follows from [21, Theorem 2.3].

Proposition 2.4.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2-boundary and $p, q \in (1, \infty)$. Then, the Neumann-Laplacian $-\Delta_N$ on $L^q(\Omega)$ with domain $D(\Delta_N) = W_N^{2,q}(\Omega)$ has a bounded $\mathcal{H}^\infty$-calculus with $\phi_{-\Delta_N}^\infty = 0$. In particular, Δ_N has the property of maximal L^p-regularity.*

Next, we give a result concerning higher regularity which is a consequence of [71, Theorem 6.3.3.]. To be more precise, we consider the Neumann-Lapacian on $W^{1,q}(\Omega)$ instead of $L^q(\Omega)$.

Proposition 2.4.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^3-boundary and $p, q \in (1, \infty)$. Then, the Neumann-Laplacian Δ_N^1 on $W^{1,q}(\Omega)$ with domain $D(\Delta_N^1) = W_N^{3,q}(\Omega)$ has the property of maximal L^p-regularity.*

Finally, as described in the proof of [44, Theorem 2.4] the maximal regularity result holds also in the $W^{-1,q}(\Omega)$-setting.

Proposition 2.4.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2-boundary and $p, q \in (1, \infty)$. Then, the Neumann-Laplacian $\Delta_{N,w}$ on $(W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'$ with domain $D(\Delta_{N,w}) = W^{1,q}(\Omega) \cap L^q_{av}(\Omega)$ has the property of maximal L^p-regularity.*

2.4.2 The Neumann-Laplacian on Convex Domains

In this subsection, we collect some properties for the Neumann-Laplacian in bounded convex domains. First, we consider the Neumann-Laplacian Δ_N defined on $L^q(\Omega)$ with domain $D(\Delta_N) = W_N^{2,q}(\Omega)$.

Proposition 2.4.4 ([85, Theorem 6.5])**.** *Let $p \in (1, \infty)$, $1 < q \leq 2$, and let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded convex domain. Then the Neumann-Laplacian Δ_N with domain $D(\Delta_N) = W_N^{2,q}(\Omega)$ has maximal L^p-regularity in $L^q(\Omega)$.*

Furthermore, we state that $-\Delta_N$ admits a bounded $\mathcal{H}^\infty$-calculus on $L^q(\Omega)$.

Proposition 2.4.5. *Let $1 < q \leq 2$, and let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded convex domain. Then the Neumann-Laplacian $-\Delta_N$ with domain $D(\Delta_N) = W_N^{2,q}(\Omega)$ has a bounded $\mathcal{H}^\infty$-calculus on $L^q(\Omega)$ of angle $\phi < \pi/2$.*

Proof. For $1 < q \leq 2$, the Neumann-Laplacian Δ_N is the generator of a positive contraction semigroup on $L^q(\Omega)$ ([85, Theorem 5.4 and Lemma 5.11]). Hence, $-\Delta_N$ has a bounded $\mathcal{H}^\infty$-calculus on $L^q(\Omega)$ of some angle $\phi > \pi/2$ ([40, Corollary 1]).

Furthermore, Δ_N in $L^2(\Omega)$ is self-adjoint. Thus, $-\Delta_N$ has a bounded $\mathcal{H}^\infty$-calculus of angle $\phi = 0$ on $L^2(\Omega)$ [35, Corollary 7.1.6]. Then, complex interpolation yields the existence of a bounded $\mathcal{H}^\infty$-calculus on $L^q(\Omega)$ of angle $\phi = \pi|1/q - 1/2| < 1/2$. $\square$

Next, we consider the weak Neumann-Laplacian $\Delta_{N,w}$ on $(W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'$ with domain $D(\Delta_{N,w}) = W^{1,q}(\Omega) \cap L^q_{av}(\Omega)$. As in the $L^q(\Omega)$-setting we show that $\Delta_{N,w}$ has a bounded $\mathcal{H}^\infty$-calculus and the maximal regularity property in the $W^{-1,q}(\Omega)$ setting.

Proposition 2.4.6. *For $p \in (1, \infty)$, $1 < q \leq 2$, and for all bounded convex domains $\Omega \subset \mathbb{R}^d$, $d \geq 3$, the weak Neumann-Laplacian $-\Delta_{N,w}$ with domain $D(\Delta_{N,w}) = W^{1,q}(\Omega) \cap L^q_{av}(\Omega)$ has a bounded $\mathcal{H}^\infty$-calculus on $(W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'$ of angle $\phi < \pi/2$. In particular, $\Delta_{N,w}$ has the property of maximal L^p-regularity.*

Proof. First, note that [87, Theorem 1.6] implies, that the domain of $\Delta_{N,w}$ on $(W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'$ is indeed $D(\Delta_{N,w}) = W^{1,q}(\Omega) \cap L^q_{av}(\Omega)$. Let Δ_N be the Neumann-Laplacian on $L^q_{av}(\Omega)$ with domain $D(\Delta_N) = W^{2,q}_N(\Omega) \cap L^q_{av}(\Omega)$. Since $-\Delta_N$ has a bounded H^∞-calculus on $L^q_{av}(\Omega)$, with Proposition 2.3.1 we have that $D(\Delta_N^{1/2}) = W^{1,q}(\Omega) \cap L^q_{av}(\Omega)$. Therefore, it follows that $D(\Delta_{N,w}^{1/2}) = L^q_{av}(\Omega)$. In order to show that $-\Delta_{N,w} \in \mathcal{H}^\infty((W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))')$ with $\mathcal{H}^\infty$-angle $\phi^\infty_{-\Delta_{N,w}} < \pi/2$, let $f \in \mathcal{H}^\infty_0$. Then, we obtain

$$
\begin{aligned}
\|f(\Delta_{N,w})\|_{\mathcal{L}((W^{1,q'}(\Omega)\cap L^{q'}_{av}(\Omega))')} &\leq \|\Delta_{N,w}^{1/2}\Delta_{N,w}^{-1/2} f(\Delta_{N,w})\|_{\mathcal{L}((W^{1,q'}(\Omega)\cap L^{q'}_{av}(\Omega))')} \\
&\leq \|\Delta_{N,w}^{1/2} f(\Delta_{N,w})\Delta_{N,w}^{-1/2}\|_{\mathcal{L}((W^{1,q'}(\Omega)\cap L^{q'}_{av}(\Omega))')} \\
&\leq c\|f(\Delta_{N,w})\Delta_{N,w}^{-1/2}\|_{\mathcal{L}((W^{1,q'}(\Omega)\cap L^{q'}_{av}(\Omega))', L^q_{av}(\Omega))} \\
&\leq c\|f(\Delta_{N,w})\|_{\mathcal{L}(L^q_{av}(\Omega))} \\
&\leq c\|f\|_{\mathcal{H}^\infty}.
\end{aligned}
$$

In the last step, we used that the Neumann-Laplacian has a bounded $\mathcal{H}^\infty$-calculus on $L^q_{av}(\Omega)$ of angle $\phi < \pi/2$ due to Proposition 2.4.5. $\square$

We close this subsection with the following result, which implies that the Neumann-Laplacian on $W^{-1,q}(\Omega) = (W^{1,q'}(\Omega))'$ has 0 as a semi-simple eigenvalue.

Proposition 2.4.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and assume that $\mu \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ is an elliptic coefficient function, i.e., there exists an $a > 0$ such that*

$$a|\xi|^2 \leq \langle \mu(x)\xi, \xi\rangle$$

for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^d$. For $q > 2$ let $-\operatorname{div}\mu\nabla$ be the $W^{-1,q}(\Omega)$ realization of the operator which is defined on $W^{-1,2}(\Omega)$ by

$$\langle -\operatorname{div}\mu\nabla\psi, \varphi\rangle = \int_\Omega \mu\nabla\psi \cdot \nabla\overline{\varphi}, \qquad \text{for } \psi, \varphi \in W^{1,2}(\Omega).$$

Suppose that the resolvent of $-\operatorname{div}\mu\nabla$ is compact. Let C denote the set of constant functions and $W_\perp^{-1,q}(\Omega)$ be the subset of $W^{-1,q}(\Omega)$ which annihilates the constants. Then, $W^{-1,q}(\Omega)$ splits up in the direct sum $C \oplus W_\perp^{-1,q}(\Omega)$. Moreover, $W_\perp^{-1,q}(\Omega)$ coincides with the range of $-\operatorname{div}\mu\nabla$ in $W^{-1,q}(\Omega)$.

Proof. First, the assumption that the resolvent is compact implies that the spectrum of $-\operatorname{div}\mu\nabla$ consists of isolated eigenvalues with finite multiplicity only ([24, Corollary 1.19]). Next, we show that the geometric and algebraic multiplicity of the eigenvalue 0 are both 1.

We start with the geometric multiplicity. If $-\operatorname{div}\mu\nabla\psi = 0$ it follows by definition that $\int_\Omega \mu\nabla\psi \cdot \nabla\overline{\psi} = 0$. Together with the assumed ellipticity of the coefficient function μ this implies that $\nabla\psi = 0$ almost everywhere on Ω. Thus, ψ has to be a constant and the eigenspace of the eigenvalue 0 consists exactly of the constant functions.

Next, assume that the algebraic multiplicity of the eigenvalue 0 is larger than 1. If this is the case, then there exists a function φ such that $-\operatorname{div}\mu\nabla\varphi = 1$. Note that this equation does not only hold in $W^{-1,q}(\Omega)$ but also in $W^{-1,2}(\Omega)$. Hence, testing with the constant function 1 yields

$$\langle 1, 1\rangle = \langle -\operatorname{div}\mu\nabla\varphi, 1\rangle = \int_\Omega \mu\nabla\varphi \cdot \nabla 1 = 0.$$

The left-hand side of this equation is nonzero, thus we have a contradiction and the algebraic multiplicity of the eigenvalue 0 is also 1. Finally, considering the spectral projection P corresponding to the eigenvalue 0 and applying [51, Chapter III.6.4] yields the desired result. □

2.4.3 The Stokes Operator

We introduce and collect some results concerning the Helmholtz projection $\mathbb{P}$ and the Stokes operator A_D. For more information, we refer to the recent survey article from Hieber and Saal [43]. Let $q \in (1, \infty)$ and $\Omega \subset \mathbb{R}^d$ be an open set. Then we define the space of solenoidal functions by

$$L^p_\sigma(\Omega) := \overline{\{u \in C_c^\infty(\overline{\Omega}) : \operatorname{div} u = 0\}}^{\|\cdot\|_{L^p}}.$$

Furthermore, we set

$$G^q(\Omega) = \{u \in L^q(\Omega) : u = \nabla\pi \text{ for some } \pi \in W^{1,q}(\Omega)\}.$$

We say that the *Helmholtz decomposition for* $L^q(\Omega)$ exists whenever $L^q(\Omega)$ can be decomposed into

$$L^q(\Omega) = L^q_\sigma(\Omega) \oplus G^q(\Omega).$$

In this case, we call the unique projection $\mathbb{P} : L^q(\Omega) \to L^q_\sigma(\Omega)$ which has $G^q(\Omega)$ as its null space *Helmholtz projection*. The Helmholtz projection is known to exists for a large class of domains, for example if Ω is a bounded C^1-domain.

After these preparations, we can state the following result on the Stokes operator $A_D = \mathbb{P}\Delta_D$, where Δ_D denotes the Dirichlet-Laplacian on $L^q(\Omega)$, see, e.g., [43, Theorem 9].

Proposition 2.4.8. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded domain with C^3-boundary and let $p, q \in (1, \infty)$. Then the Stokes operator*

$$A_D := \mathbb{P}\Delta u, \quad D(A_D) = \mathrm{W}^{2,q}(\Omega) \cap \mathrm{W}^{1,q}_0(\Omega) \cap \mathrm{L}^q_\sigma(\Omega)$$

admits maximal L^p-regularity on $\mathrm{L}^q_\sigma(\Omega)$. Moreover, $\sigma(A_D) = (-\infty, -\kappa]$ for some $\kappa = \kappa(\Omega) > 0$, i.e., A_D is invertible.

2.5 Applications

This section shows some applications of the abstract operator theory introduced before in this chapter. First, we provide some time-space embeddings which arise from the mixed derivative theorem. After that we recall the theory for quasilinear parabolic problems with initial values followed by the recent theory for the periodic setting. In both settings it will be essential that the involved linear operator satisfies the maximal regularity property.

2.5.1 Time-Space Embeddings

We collect some time-space embeddings, which will be used frequently in the following chapters to estimate nonlinear terms of the respective equations. These embeddings follow from the classical mixed derivative theorem, see, e.g., [71, Corollary 4.5.10] or [72, Remark 1.1]. The embeddings then follow by using the results from Section 2.4.

First, we consider embeddings on bounded C^3-domains.

Proposition 2.5.1. *Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ be a bounded C^3-domain. Let $T > 0$, $p, q \in (1, \infty)$, and $\theta \in [0, 1]$. Then the following continuous embeddings are valid*

$$
\begin{aligned}
L^p(0,T; W^{1,q}(\Omega) \cap L^q_{av}(\Omega)) \cap W^{1,p}(0,T; (W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))')& \\
\hookrightarrow H^{\theta,p}(0,T; H^{-1+2(1-\theta),q}(\Omega))&,\\
L^p(0,T; W^{2,q}_N(\Omega)) \cap W^{1,p}(0,T; L^q(\Omega)) \hookrightarrow H^{\theta,p}(0,T; H^{2(1-\theta),q}(\Omega))&,\\
L^p(0,T; W^{3,q}_N(\Omega)) \cap W^{1,p}(0,T; W^{1,q}(\Omega)) \hookrightarrow H^{\theta,p}(0,T; H^{1+2(1-\theta),q}(\Omega))&,\\
L^p(0,T; D(A_D)) \cap W^{1,p}(0,T; L^q_\sigma(\Omega)) \hookrightarrow H^{\theta,p}(0,T; H^{2(1-\theta),q}(\Omega))&.
\end{aligned}
$$

Similar, for convex domains we obtain the following proposition.

Proposition 2.5.2. *Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ be a bounded convex domain. Let $T > 0$, $p \in (1, \infty)$, $1 < q \leq 2$, and $\theta \in [0, 1]$. Then the following continuous embeddings are valid*

$$
\begin{aligned}
L^p(0,T; W^{1,q}(\Omega) \cap L^q_{av}(\Omega)) \cap W^{1,p}(0,T; (W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))')& \\
\hookrightarrow H^{\theta,p}(0,T; H^{-1+2(1-\theta),q}(\Omega))&,\\
L^p(0,T; W^{2,q}_N(\Omega)) \cap W^{1,p}(0,T; L^q(\Omega)) \hookrightarrow H^{\theta,p}(0,T; H^{2(1-\theta),q}(\Omega))&
\end{aligned}
$$

2.5.2 Quasilinear Parabolic Equations with Initial Value

This subsection is devoted to collect some results on abstract quasilinear parabolic problems of the form

$$\begin{cases} u'(t) - A(u(t))u(t) = F(u(t)), & t \in (0,\infty), \\ u(0) = u_0. \end{cases} \tag{2.4}$$

This theory is described, e.g., in the monograph by Prüss and Simonett [71, Chapter 5], see also [41], [53], and [60].

Let X_0 and X_1 be Banach spaces with X_1 densely embedded in X_0. For $p \in (1,\infty)$ and $\mu \in (1/p, 1]$ put $X_{\gamma,\mu} := (X_0, X_1)_{\mu-1/p,p}$. Let V_μ be an open subset of this real interpolation space and $J = (0,T)$ be a time interval for some $T > 0$. To see and exploit the effect of parabolic regularization in the L^p-framework, it is useful to consider so called time-weighted spaces. Hence, we define for a Banach space X the spaces

$$\begin{aligned} L^p_\mu(J;X) &:= \{u : J \to X : t^{1-\mu}u \in L^p(J;X)\}, \\ H^{1,p}_\mu(J;X) &:= \{u \in L^p_\mu(J;X) \cap H^{1,1}(J;X) : u' \in L^p_\mu(J;X)\}. \end{aligned}$$

Now, we define the space of time-weighted maximal L^p_μ-regularity as

$$\mathbb{E}_\mu := H^{1,p}_\mu(J;X_0) \cap L^p_\mu(J;X_1).$$

The trace space for this class of functions is given by $X_{\gamma,\mu}$ and the corresponding data space by $\mathbb{F}_\mu := L^p_\mu(J;X_0)$. Note that the choice $\mu = 1$ yields the classical non-weighted spaces. When considering these spaces, we omit the μ in the notation, e.g., we write $L^p(J;X) := L^p_1(J;X)$. In the following approach, it is essential that the operators A(u) have the property of maximal L^p-regularity as defined in Section 2.2.

If an operator A_0 in X_0 with domain X_1 has maximal L^p- regularity, it is known that maximal regularity also holds in the time-weighted case, see [70, Theorem 2.4]. The result on local well-posedness to (2.4) reads as follows.

Proposition 2.5.3 ([71, Theorem 5.1.1])**.** *Let $p \in (1,\infty)$, $u_0 \in V_\mu$ be given and suppose that (A,F) satisfy*

$$(A,F) \in C^{1-}(V_\mu; \mathcal{L}(X_1,X_0) \times X_0),$$

for some $\mu \in (1/p, 1]$, i.e., A and F are locally Lipschitz continuous. Assume in addition that $A(u_0)$ admits maximal L^p-regularity.

Then there exists $T = T(u_0) > 0$ and $r = r(u_0) > 0$ with $\overline{B_{X_{\gamma,\mu}}}(u_0, r) \subset V_\mu$ such that (2.4) admits a unique solution

$$u = u(\cdot, u_1) \in \mathbb{E}_\mu \cap C([0, T]; V_\mu),$$

on $[0, T]$ for any intial value $u_1 \in \overline{B_{X_{\gamma,\mu}}}(u_0, r)$. There exists a constant $c = c(u_0) > 0$ such that for all $u_1, u_2 \in \overline{B_{X_{\gamma,\mu}}}(u_0, r)$ the estimate

$$\|u(\cdot, u_1) - u(\cdot, u_2)\|_{\mathbb{E}} \leq c\|u_1 - u_2\|_{X_{\gamma,\mu}}$$

is valid. In addition , for each $\delta \in (0, T)$ we have

$$u \in H^{1,p}(\delta, T; X_0) \cap L^p(\delta, T; X_1) \hookrightarrow C([\delta, T]; X_\gamma),$$

i.e., the solution regularizes instantaneously.

Next, we state a result on convergence of solutions starting near an equilibrium $u_\star$ of (2.4), which is known as *generalized principle of linearized stability*. In order to do so, assume there exists an open set $V \subset X_\gamma$ such that

$$(A, F) \in C^1(V, \mathcal{L}(X_1, X_0) \times X_0). \tag{2.5}$$

Let $\mathcal{E} \subset V \cap X_1$ denote the set of equilibrium solutions of (2.4), that is

$$u \in \mathcal{E} \quad \text{if and only if} \quad u \in V \cap X_1, A(u)u = F(u).$$

Given an element $u_\star \in \mathcal{E}$, we assume that $u_\star$ is contained in an m-dimensional *manifold of equilibria*. This means that there is an open subset $U \subset \mathbb{R}^m$ with $0 \in U$ and a C^1-function $\Psi : U \to X_1$ such that

- $\Psi(U) \subset \mathcal{E}$ and $\Psi(0) = u_\star$,
- the rank of $\Psi'(0)$ equals m,
- $A(\Psi(\zeta))\Psi(\zeta) = F(\zeta), \quad \zeta \in U.$

Suppose that the operator $A(u_\star)$ has maximal L^p-regularity and define the full linearization of (2.4) at $u_\star$ by

$$A_0 w = A(u_\star)w + (A'(u_\star)w)u_\star - F'(u_\star)w \quad \text{for } w \in X_1. \tag{2.6}$$

After these preparations, the result on convergence of solutions starting near $u_\star$ reads as follows.

Proposition 2.5.4 ([73, Theorem 2.1])**.** *Let $1 < p < \infty$. Suppose $u_\star \in V \cap X_1$ is an equilibrium of* (2.4)*, and suppose that the functions (A, F) satisfy* (2.5)*. Suppose further that $A(u_\star)$ has the property of maximal L^p-regularity and let A_0 be defined as in* (2.6)*. Suppose that $u_\star$ is normally stable, i.e.,*

(i) *near $u_\star$ the set of equilibria $\mathcal{E}$ is a C^1-manifold in X_1 of dimension $m \in \mathbb{N}$,*

(ii) *the tangent space of $\mathcal{E}$ at $u_\star$ is isomorphic to $\mathrm{N}(A_0)$,*

(iii) 0 *is a semi-simple eigenvalue of A_0, i.e., $\mathrm{N}(A_0) \oplus \mathrm{R}(A_0) = X_0$,*

(iv) $\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Then $u_\star$ is stable in X_1. Furthermore, there exists a number $\delta > 0$ such that the unique solution $u(t)$ of (2.4) *with initial value $u_0 \in B_{X_\gamma}(u_\star, \delta)$ exists on $\mathbb{R}_+$ and converges at an exponential rate in X_γ to some $u_\infty \in \mathcal{E}$ as $t \to \infty$.*

2.5.3 Time-Periodic Quasilinear Parabolic Equations

In this subsection, we introduce the abstract theory for time-periodic solutions to semilinear and quasilinear parabolic evolution equations. The theory is based on the articles by Arendt and Bu [7], who considered maximal periodic L^p-regularity for the linear problem as defined in Section 2.2, and by Hieber and Stinner [44], who combined the linear theory with the contraction mapping principle for the semilinear and quasilinear setting.

Let X_0 and X_1 be Banach spaces with X_1 densely embedded in X_0 and $A : X_1 \to X_0$ be a closed linear operator. Let $p \in (1, \infty)$ and set

$$\mathbb{F} := L^p(0, 2\pi; X_0), \qquad \mathbb{E} := H^{1,p}(0, 2\pi; X_0) \cap L^p(0, 2\pi; X_1). \tag{2.7}$$

Next, we consider the time-periodic semilinear problem

$$
\begin{cases} u'(t) - Au(t) = F(t, u(t)), & t \in (0, 2\pi), \\ u(0) = u(2\pi). \end{cases} \tag{2.8}
$$

For $p \in (1, \infty)$ define $X_\gamma := (X_0, X_1)_{1-1/p,p}$. Recalling the definition of $\mathbb{E}$ and $\mathbb{F}$ from (2.7), it is well known (see e.g. [4, Theorem 4.10.2 in Chapter III]) that

$$
\mathbb{E} \hookrightarrow BUC([0, 2\pi]; X_\gamma).
$$

We assume the following Lipschitz condition on the right-hand side F.

(L) Let $F : [0, 2\pi] \times X_\gamma \to X_0$ satisfy $F(\cdot, v(\cdot)) \in \mathbb{F}$ for all $v \in \mathbb{E}$ und suppose that for each $R > 0$ there exists a $C_R > 0$ such that

$$
\|F(\cdot, v(\cdot)) - F(\cdot, w(\cdot))\|_{\mathbb{F}} \leq C_R \|v - w\|_{\mathbb{E}}
$$

for all $v, w \in \overline{B_{\mathbb{E}}}(0, R)$.

Then, the result on existence of periodic solutions to the abstract semilinear system (2.8) reads as follows.

Proposition 2.5.5 ([44, Corollary 3.5])**.** *Let $A : X_1 \to X_0$ be a linear operator satisfying maximal periodic L^p-regularity for $p \in (1, \infty)$. Assume that Assumption (L) is satisfied.*
Then there is $\delta_1 > 0$ such that, if $C_R < \delta_1$ for some $R > 0$, then there are $\delta_2 > 0$ and $r > 0$ such that if $\|F(\cdot, 0)\|_{\mathbb{F}} < \delta_2$ there is a unique periodic solution $u \in \overline{B_{\mathbb{E}}}(0, r)$ to (2.8)*.*

Considering the quasilinear time-periodic problem

$$
\begin{cases} u'(t) - A(u(t))u(t) = F(t, u(t)), & t \in (0, 2\pi), \\ u(0) = u(2\pi), \end{cases} \tag{2.9}
$$

and using the notation as before, we additionally need the following assumption on the family of linear operators.

(Q) Let $A : X_\gamma \to \mathcal{L}(X_1, X_0)$ be a family of closed linear operators and suppose that for each $R > 0$ there exists $L(R) > 0$ such that

$$\|A(v(\cdot))w(\cdot) - A(\overline{v}(\cdot))w(\cdot)\|_{\mathbb{F}} \leq L(R)\|v - \overline{v}\|_{\mathbb{E}}\|w\|_{\mathbb{E}}$$

for all v, $\overline{v}$, $w \in \overline{B_{\mathbb{E}}}(0, R)$.

Then, the result on existence of time-periodic solutions to the abstract quasilinear system (2.9) reads as follows.

Proposition 2.5.6 ([44, Theorem 3.3]). *Let assumptions (L) and (Q) be satisfied and assume that $A(0)$ admits maximal periodic L^p-regularity for $p \in (1, \infty)$.*
Then there is $\delta_1 > 0$ such that, if $C_R < \delta_1$ for some $R > 0$, then there are $\delta_2 > 0$ and $r > 0$ such that if $\|F(\cdot, 0)\|_{\mathbb{F}} < \delta_2$ there is a unique periodic solution $u \in \overline{B_{\mathbb{E}}}(0, r)$ to (2.9).

Chemotaxis

CHAPTER 3

A Coupled Chemotaxis-Navier–Stokes System on Smooth Domains

In this chapter as well as in the following, we consider systems describing chemotaxis. Chemotaxis is the direct movement of cells and organisms (e.g. bacteria) in response to chemical gradients. A typical model describing these phenomena is the Keller–Segel model for chemotaxis, which was first introduced by Keller and Segel in 1970 [52]. Their originally proposed form of the model consisted of four coupled reaction-advection-diffusion equations. But in the same paper Keller and Segel reduced their system under quasi-steady-state assumptions to a model consisting of two coupled parabolic equations for two unknown functions. We will consider the *classical* Keller–Segel system of the form

$$\text{(KS)} \qquad \begin{cases} \partial_t n = \Delta n - \nabla \cdot (n \nabla c) & \text{in } (0,\infty) \times \Omega, \\ \partial_t c = \Delta c - c + n & \text{in } (0,\infty) \times \Omega, \\ \dfrac{\partial n}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = 0 & \text{on } (0,\infty) \times \partial\Omega. \end{cases}$$

Here n and c denote the cell (or organism) density and the concentration of the chemical signal (e.g. oxygen concentration), respectively. The physical domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is supposed to be bounded and ν denotes the outward unit normal vector on $\partial\Omega$.

There are many results concerning local and global existence as well as blow-up of solutions for different versions of the Keller–Segel model. For details we refer to the survey articles [45, 46] and the references therein. Recently, Hieber and Stinner [44] proved the existence and uniqueness of strong time-periodic solutions for the Keller–Segel model on smooth domains. We want to extend their results in two different directions. In Chapter 4 we lower the regularity of the domain, namely we consider the Keller–Segel model (KS) on bounded convex domains. In this chapter, we consider a system which couples chemotaxis and the motion of fluids. This coupling describes that cells and chemical substrates are transported by the surrounding fluid and that the motion of the fluid is under the influence of the gravitational forcing generated by aggregation of cells.

To be more precise, we study the following coupled chemotaxis-Navier–Stokes system

$$\text{(KSNS)} \qquad \begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) & \text{in } (0,\infty) \times \Omega, \\ \partial_t c + u \cdot \nabla c = \Delta c - c + n & \text{in } (0,\infty) \times \Omega, \\ \partial_t u - \Delta u - \nabla P = \kappa (u \cdot \nabla) u + n \nabla \phi & \text{in } (0,\infty) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0,\infty) \times \Omega, \\ \dfrac{\partial n}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega. \end{cases}$$

Here n and c denote the bacterial density and the oxygen concentration, whereas u denotes the fluid velocity and P the associated pressure. The transport of the cells and chemical substrates is described by $u \cdot \nabla n$ and

$u \cdot \nabla c$. The physical domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is supposed to be a bounded domain with smooth boundary. Furthermore, $\kappa \in \mathbb{R}$ is a fixed number which distinguishes between the Stokes and the Navier–Stokes case, ν denotes the outward unit normal vector on $\partial\Omega$, and the gravitational forcing $\nabla\phi$ is a bounded function.

This chapter is structured as follows. In Section 3.1 we show the existence and uniqueness of strong time-periodic solutions to the coupled chemotaxis-Navier–Stokes equations. For this purpose, we rewrite the equations as abstract evolution equations and apply the theory introduced in Subsection 2.5.3. Afterwards, in Section 3.2 we consider the initial-value problem and prove local well-posedness.

3.1 The Time-Periodic Problem

This section is devoted to the existence and uniqueness of a strong time-periodic solution to the periodic chemotaxis-Navier–Stokes system

(PKSNS)

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) + f_1(t) & \text{in } (0,\infty) \times \Omega, \\ \partial_t c + u \cdot \nabla c = \Delta c - c + n + f_2(t) & \text{in } (0,\infty) \times \Omega, \\ \partial_t u - \Delta u - \nabla P = \kappa (u \cdot \nabla) u + n \nabla \phi + f_3(t) & \text{in } (0,\infty) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0,\infty) \times \Omega, \\ \dfrac{\partial n}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ (n,c,u)(x,0) = (n,c,u)(x,T) & \text{in } \Omega, \end{cases}$$

provided the T-periodic external forces f_1, f_2, and f_3 are sufficiently small, where $T > 0$ is a prescribed time period. Since n and c denote the bacterial density and the oxygen concentration, respectively, $n, c \geq 0$ is a natural assumption and many studies for the Keller–Segel system are dealing with the latter. Recall that the abstract approach by Hieber and Stinner to the classical Keller-Segel system introduced in Subsection 2.5.3, which we want to apply to the coupled system, needs the involved linear operator to be invertible. Hence, the operators describing the diffusion processes have to be invertible. But 0 is the first eigenvalue of the Neumann-Laplacian and in order to handle this, the first solution component n needs to satisfy the condition $\int_\Omega n(x,t)\ \mathrm{d}x = 0$ for any $t \geq 0$. To guarantee the latter, we will transform (PKSNS) into a slightly different form.

Therefore, given T-periodic functions f_1, f_2, and f_3 we assume that (N, C, u) is a T-periodic solution to (PKSNS) with (N, C) nonnegative. In particular, $N(\cdot, 0)$ and $C(\cdot, 0)$ are nonnegative at time $t = 0$. Having the comparison principle in mind, we see that the nonnegativity of N and C can only be guaranteed if f_1 and f_2 are nonnegative. Hence, assume that f_1 and f_2 are nonnegative. Then, integrating the first equation of (PKSNS) and using the Neumann and Dirichlet boundary conditions in combination with the divergence theorem we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega N(x,t)\ \mathrm{d}x = \int_\Omega f_1(t)\ \mathrm{d}x = |\Omega| f_1(t)$$

for any $t > 0$. Next, we set $M(t) := \frac{1}{|\Omega|}\int_\Omega N(x,t)\ \mathrm{d}x$, which thus satisfies $M(t) = M(0) + \int_0^t f_1(t)\ \mathrm{d}t$. The T-periodicity of N yields $M(T) = M(0)$ and the latter implies $\int_0^T f_1(t)\ \mathrm{d}t = 0$. As f_1 is nonnegative, this yields $f_1 \equiv 0$ and therefore $M(t) = M(0) =: M$ is a constant for all $t \geq 0$.

Next, we define $n(x,t) := N(x,t) - M$ and $c(x,t) := C(x,t) - M$. Since $f_1 \equiv 0$ and $M \geq 0$, we obtain $\int_\Omega n(x,t)\ \mathrm{d}x = 0$ for all $t \geq 0$ and (n, c, u) is a solution to

(PKSNS II)

$$\begin{cases} \partial_t n + u\cdot\nabla n = \Delta n - \nabla\cdot((n+M)\nabla c) & \text{in } (0,\infty)\times\Omega, \\ \partial_t c + u\cdot\nabla c = \Delta c - c + n + f_2(t) & \text{in } (0,\infty)\times\Omega, \\ \partial_t u - \Delta u - \nabla P = \kappa(u\cdot\nabla)u + (n+M)\nabla\phi + f_3(t) & \text{in } (0,\infty)\times\Omega, \\ \nabla\cdot u = 0 & \text{in } (0,\infty)\times\Omega, \\ \dfrac{\partial n}{\partial\nu} = \dfrac{\partial c}{\partial\nu} = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ (n,c,u)(x,0) = (n,c,u)(x,T) & \text{in } \Omega. \end{cases}$$

We show the existence of a (possibly negative) time-periodic solution to this system. Then, we use this solution to deduce the existence of a time-periodic solution to (PKSNS) for which the first two components are nonnegative.

Afterwards, we consider case that the quasilinear Keller–Segel model with non-degenerate diffusion is coupled with the Navier–Stokes system. Namely, we consider the periodic system

(PQKSNS)
$$\begin{cases} \partial_t n + u\cdot\nabla n = \nabla\cdot((n+1)^m\nabla n) & \\ \qquad\qquad - \nabla\cdot(n\nabla c) + f_1(t) & \text{in } (0,\infty)\times\Omega,\\ \partial_t c + u\cdot\nabla c = \Delta c - c + n + f_2(t) & \text{in } (0,\infty)\times\Omega,\\ \partial_t u - \Delta u = \kappa(u\cdot\nabla)u + \nabla P + n\nabla\phi + f_3(t) & \text{in } (0,\infty)\times\Omega,\\ \nabla\cdot u = 0 & \text{in } (0,\infty)\times\Omega,\\ \dfrac{\partial n}{\partial\nu} = \dfrac{\partial c}{\partial\nu} = 0 & \text{on } (0,\infty)\times\partial\Omega,\\ u = 0 & \text{on } (0,\infty)\times\partial\Omega,\\ (n,c,u)(x,0) = (n,c,u)(x,T) & \text{in } \Omega, \end{cases}$$

where $m\in\mathbb{R}$ is a fixed number. The rest of the notation is as in the semilinear setting. For the quasilinear case we proceed similarly to above. Let (N,C,u) be a T-periodic solution to (PQKSNS) with $f_1 = 0$, $f_2 \geq 0$, and (N,C) nonnegative. By an argument as in the semilinear case $M(t) = \frac{1}{|\Omega|}\int_\Omega N(x,t)\,\mathrm{d}x$ is a constant, and $n(x,t) := N(x,t) - M$ and $c(x,t) := C(x,t) - M$ satisfy $\int_\Omega n(x,t)\,\mathrm{d}x = 0$ for all $t\geq 0$ and (n,c,u) is a solution to

(PQKSNS II)
$$\begin{cases} \partial_t n + u\cdot\nabla n = \nabla\cdot((n+M+1)^m\nabla n) & \\ \qquad\qquad - \nabla\cdot((n+M)\nabla c) & \text{in } (0,\infty)\times\Omega,\\ \partial_t c + u\cdot\nabla c = \Delta c - c + n + f_2(t) & \text{in } (0,\infty)\times\Omega,\\ \partial_t u - \Delta u - \nabla P = \kappa(u\cdot\nabla)u + (n+M)\nabla\phi + f_3(t) & \text{in } (0,\infty)\times\Omega,\\ \nabla\cdot u = 0 & \text{in } (0,\infty)\times\Omega,\\ \dfrac{\partial n}{\partial\nu} = \dfrac{\partial c}{\partial\nu} = 0 & \text{on } (0,\infty)\times\partial\Omega,\\ u = 0 & \text{on } (0,\infty)\times\partial\Omega,\\ (n,c,u)(x,0) = (n,c,u)(x,T) & \text{in } \Omega. \end{cases}$$

In order to prove the existence of a T-periodic solution to both the semilinear and quasilinear system, which is unique in the associated maximal regularity space, in the following subsections we will rewrite (PKSNS II) as a semilinear evolution equation and (PQKSNS II) as a quasilinear evolution equation in the L^p-setting. Then, we use the quasilinear version of the Arendt–Bu theorem introduced in Subsection 2.5.3 to show the existence of strong time-periodic solutions. This will be done in two different settings, the strong and the weak setting, described in Subsection 3.1.1 and Subsection 3.1.2, respectively.

3.1.1 Strong Setting

First, we focus on the strong setting. To this end, recall for $q \in (1, \infty)$ and $k \in \{2, 3\}$ the spaces

$$L^q_{av}(\Omega) = \{g \in L^q(\Omega) : \int_\Omega g \,\mathrm{d}x = 0\},$$
$$W^{k,q}_N(\Omega) = \{g \in W^{k,q}(\Omega) : \frac{\partial g}{\partial \nu} = 0 \text{ on } \partial\Omega\}.$$

Here, the underlying ground space X_0 is given by

$$X_0 := L^q_{av}(\Omega) \times W^{1,q}(\Omega) \times L^q_\sigma(\Omega).$$

In order to apply the abstract theory introduced in Subsection 2.5.3 to the coupled chemotaxis-Navier–Stokes system (PKSNS II), we have to rewrite the system. For this purpose, we define on X_0 the operator $\mathcal{A}$ and the mapping F as

$$\mathcal{A} := \begin{pmatrix} \Delta_N & 0 & 0 \\ 1 & \Delta_N^1 - 1 & 0 \\ \mathbb{P}\nabla\phi & 0 & A_D \end{pmatrix},$$
$$F(t, w) := \begin{pmatrix} -\nabla \cdot ((n + M)\nabla c) - u \cdot \nabla n \\ -u \cdot \nabla c + f_2(t) \\ \mathbb{P}[\kappa(u \cdot \nabla)u] + \mathbb{P}[M\nabla\phi] + \mathbb{P}f_3(t) \end{pmatrix}$$

for $w = (n, c, u)^T$, where Δ_N with domain $D(\Delta_N) = W^{2,q}_N(\Omega) \cap L^q_{av}(\Omega)$ denotes the Neumann–Laplacian on $L^q_{av}(\Omega)$, and Δ_N^1 with domain $D(\Delta_N^1) =$

$W^{3,q}_N(\Omega)$ denotes the Neumann–Laplacian on $W^{1,q}(\Omega)$. Furthermore, for Δ_D being the Dirichlet-Laplacian on $L^q(\Omega)$, we denote the Stokes operator on $L^q_\sigma(\Omega)$ by $A_D = \mathbb{P}\Delta_D$ with domain $D(A_D) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$. Then, using this notation and as usual the Helmholtz projection $\mathbb{P}$, we may rewrite equation (PKSNS II) in the periodic setting as

$$\begin{cases} \partial_t w(t) - \mathcal{A}w(t) = F(t, w(t)) & t \in (0,T), \\ \qquad\qquad w(0) = w(T). \end{cases} \tag{3.1}$$

Furthermore, for $p, q \in (1,\infty)$ we define the data space $\mathbb{F} := L^p(0,T; X_0)$ and the solution spaces for n, c, and u by

$$\begin{aligned} \mathbb{E}_1 &:= L^p(0,T; W^{2,q}_N(\Omega) \cap L^q_{av}(\Omega)) \cap W^{1,p}(0,T; L^q_{av}(\Omega)), \\ \mathbb{E}_2 &:= L^p(0,T; W^{3,q}_N(\Omega)) \cap W^{1,p}(0,T; W^{1,q}(\Omega)), \\ \mathbb{E}_3 &:= L^p(0,T; W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)) \cap W^{1,p}(0,T; L^q_\sigma(\Omega)), \end{aligned}$$

respectively, as well as the solution space for the whole system $\mathbb{E} := \mathbb{E}_1 \times \mathbb{E}_2 \times \mathbb{E}_3$. Provided that

$$p, q \in (1,\infty) \quad \text{such that} \quad q > \frac{d}{2} \quad \text{and} \quad \frac{1}{p} + \frac{d}{2q} < 1 \tag{3.2}$$

our result on existence of strong time-periodic solutions to (PKSNS II) in the strong setting reads as follows.

Theorem 3.1.1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with smooth boundary, $M > 0$, $T > 0$, and assume that* (3.2) *is satisfied. Let $f = (0, f_2, f_3)^T \in \mathbb{F}$ be T-periodic.*

Then there are $r_0 > 0$ and $M_0 > 0$ such that for any $r \in (0, r_0)$ there exists $\delta = \delta(r) > 0$ such that if $\|f\|_{\mathbb{F}} < \delta$ and $M < M_0$, then there exists a T-periodic solution $w = (n, c, u)^T \in \mathbb{E}$ to (PKSNS II)*, which is unique in $\overline{B_{\mathbb{E}}}(0, r)$.*

Proof. Due to Proposition 2.5.5 it suffices to show the following properties:

a) The operator $\mathcal{A}$ admits maximal periodic L^p-regularity on X_0.

b) $F(\cdot, w(\cdot)) \in \mathbb{F}$ for any $w \in \mathbb{E}$.

c) There exists a $C > 0$ such that for any $R > 0$ it holds

$$\|F(\cdot, w(\cdot)) - F(\cdot, \tilde{w}(\cdot))\|_{\mathbb{F}} \leq C(R + M)\|w - \tilde{w}\|_{\mathbb{E}}$$

for all $w, \tilde{w} \in \overline{B_{\mathbb{E}}}(0, R)$.

If a), b), and c) are satisfied, we may define C_R for $R \in (0, \frac{\delta_1}{2C})$ by $C_R := C(R+M)$. Then, choosing $M_0 := \frac{\delta_1}{2C}$, the assertion of the theorem follows from Proposition 2.5.5 for $\delta := \delta_2$ and $r_0 := \frac{\delta_1}{2C}$.

Thus, it remains to prove the assertions a), b), and c).

First, we show assertion a). For the three involved operators we have $0 \in \rho(\Delta_N)$, $0 \in \rho(\Delta_N^1 - 1)$ and $0 \in \rho(A_D)$. Hence, the triangular structure of $\mathcal{A}$ implies that $0 \in \rho(\mathcal{A})$ holds true. Moreover, as described in Section 2.4, the operators Δ_N on $L^q_{av}(\Omega)$, Δ_N^1 on $W^{1,q}(\Omega)$, and A_D on $L^q_\sigma(\Omega)$ admit maximal L^p-regularity, which implies, again due to the triangular structure, that the operator matrix $\mathcal{A}$ admits maximal L^p-regularity on X_0. Now Proposition 2.2.1 yields that $\mathcal{A}$ admits maximal periodic L^p-regularity on X_0, which proves a).

Next, we show that $F(\cdot, w(\cdot)) \in \mathbb{F}$ for any $w \in \mathbb{E}$. We estimate

$$\begin{aligned}
&\|F(\cdot, w(\cdot))\|_{\mathbb{F}} \\
&\quad \leq \|-\nabla \cdot ((n + M)\nabla c) - u \cdot \nabla n\|_{L^p(0,T;L^q_{av}(\Omega))} \\
&\qquad + \|-u \cdot \nabla c + f_2(t)\|_{L^p(0,T;W^{1,q}(\Omega))} \\
&\qquad + \|\mathbb{P}[\kappa(u \cdot \nabla)u] + \mathbb{P}[M\nabla\phi] + \mathbb{P}f_3(t)\|_{L^p(0,T;L^q_\sigma(\Omega))} \\
&\quad \leq \|\nabla n \nabla c\|_{L^p(0,T;L^q(\Omega))} + \|(n + M)\Delta c\|_{L^p(0,T;L^q(\Omega))} \\
&\qquad + \|u \cdot \nabla n\|_{L^p(0,T;L^q(\Omega))} + \|u \cdot \nabla c\|_{L^p(0,T;W^{1,q}(\Omega))} \\
&\qquad + \|\kappa(u \cdot \nabla)u\|_{L^p(0,T;L^q(\Omega))} + \|M\nabla\phi\|_{L^p(0,T;L^q(\Omega)} + \|f\|_{\mathbb{F}}.
\end{aligned}$$

By Hölder's inequality and Sobolev embeddings we obtain

$$\begin{aligned}
&\|F(\cdot, w(\cdot))\|_{\mathbb{F}} \\
&\quad \leq \|\nabla n\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\nabla c\|_{L^{2p}(0,T;L^{2q}(\Omega))} \\
&\qquad + \|n\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\Delta c\|_{L^{2p}(0,T;L^{2q}(\Omega))} + M\|\Delta c\|_{L^p(0,T;L^q(\Omega))} \\
&\qquad + \|u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\nabla n\|_{L^{2p}(0,T;L^{2q}(\Omega))} \\
&\qquad + c_0\|u\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\|\nabla c\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} \\
&\qquad + |\kappa|\|u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\nabla u\|_{L^{2p}(0,T;L^{2q}(\Omega))}
\end{aligned}$$

$$
\begin{aligned}
&+ M\|\nabla\phi\|_{L^p(0,T;L^q(\Omega))} + \|f\|_{\mathbb{F}} \\
&\le c_1\Big(\|n\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\|c\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} \\
&\quad + \|n\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|c\|_{L^{2p}(0,T;W^{2,2q}(\Omega))} + M\|c\|_{L^p(0,T;W^{3,q}(\Omega))} \\
&\quad + \|u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|n\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} \\
&\quad + \|u\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\|c\|_{L^{2p}(0,T;W^{2,2q}(\Omega))} \\
&\quad + \|u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|u\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\Big) \\
&\quad + M\|\nabla\phi\|_{L^p(0,T;L^q(\Omega))} + \|f\|_{\mathbb{F}}.
\end{aligned}
\tag{3.3}
$$

Next, we want to use the mixed derivative theorem and Sobolev embeddings. Proposition 2.5.1 yields

$$
\begin{aligned}
&\mathbb{E}_1 \hookrightarrow H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega)), \quad \mathbb{E}_2 \hookrightarrow H^{\theta,p}(0,T;H^{1+2(1-\theta),q}(\Omega)), \\
&\mathbb{E}_3 \hookrightarrow H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))
\end{aligned}
\tag{3.4}
$$

for any $\theta \in (0,1)$. Due to (3.2) we deduce the existence of $\theta \in (0,1)$ satisfying

$$
\frac{1}{2p} < \theta < \frac{1}{2} - \frac{d}{4q}.
$$

This inequality is equivalent to

$$
\theta - \frac{1}{p} > -\frac{1}{2p}, \quad 2(1-\theta) - \frac{d}{q} > 1 - \frac{d}{2q}.
$$

Hence, Sobolev embeddings imply

$$
\begin{aligned}
H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega)) &\hookrightarrow L^{2p}(0,T;W^{1,2q}(\Omega)) \hookrightarrow L^{2p}(0,T;L^{2q}(\Omega)), \\
H^{\theta,p}(0,T;H^{1+2(1-\theta),q}(\Omega)) &\hookrightarrow L^{2p}(0,T;W^{2,2q}(\Omega)) \hookrightarrow L^{2p}(0,T;W^{1,2q}(\Omega)).
\end{aligned}
\tag{3.5}
$$

Using first (3.5) and then (3.4) in (3.3), we obtain

$$
\begin{aligned}
&\|F(\cdot,w(\cdot))\|_{\mathbb{F}} \\
&\quad \le c_2\Big(\|n\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|c\|_{H^{\theta,p}(0,T;H^{1+2(1-\theta),q}(\Omega))} \\
&\qquad + \|n\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|c\|_{H^{\theta,p}(0,T;H^{1+2(1-\theta),q}(\Omega))}
\end{aligned}
$$

$$
\begin{aligned}
&+ \|u\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|n\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \\
&+ \|u\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|c\|_{H^{\theta,p}(0,T;H^{1+2(1-\theta),q}(\Omega))} \\
&+ \|u\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|u\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} + M\|c\|_{\mathbb{E}_2}\Big) \\
&+ M\|\nabla\phi\|_{L^p(0,T;L^q(\Omega))} + \|f\|_{\mathbb{F}} \\
\leq\ & c_3\Big(\|n\|_{\mathbb{E}_1}\|c\|_{\mathbb{E}_2} + \|n\|_{\mathbb{E}_1}\|c\|_{\mathbb{E}_2} + \|u\|_{\mathbb{E}_3}\|n\|_{\mathbb{E}_1} + \|u\|_{\mathbb{E}_3}\|c\|_{\mathbb{E}_2} \\
&+ \|u\|_{\mathbb{E}_3}\|u\|_{\mathbb{E}_3} + M\|c\|_{\mathbb{E}_2}\Big) + M\|\nabla\phi\|_{L^p(0,T;L^q(\Omega))} + \|f\|_{\mathbb{F}} \\
\leq\ & c_4(\|w\|_{\mathbb{E}}^2 + M\|c\|_{\mathbb{E}_2}) + M\|\nabla\phi\|_{L^p(0,T;L^q(\Omega))} + \|f\|_{\mathbb{F}}.
\end{aligned}
$$

Next, we have to show that the first component of F belongs to the space $L^p(0,T;L^q_{av}(\Omega))$. In view of the fact that u is divergence-free, the divergence theorem yields

$$
\begin{aligned}
&\int_\Omega (-\nabla\cdot((n+M)\nabla c)(\cdot,t) - (u\cdot\nabla n)(\cdot,t)\ \mathrm{d}x \\
&\quad = \int_\Omega (-\nabla\cdot((n+M)\nabla c))(\cdot,t) - (\nabla\cdot(un))(\cdot,t)\ \mathrm{d}x \\
&\quad = -\int_{\partial\Omega} ((n+M)\nabla c)(\cdot,t)\cdot\nu\ \mathrm{d}\sigma - \int_{\partial\Omega}(un)(\cdot,t)\cdot\nu\ \mathrm{d}\sigma \\
&\quad = 0
\end{aligned}
$$

for almost all $t \in (0,T)$. Note that we used that $w \in \mathbb{E}$ implies that $\frac{\partial c}{\partial\nu} = 0$ and $u = 0$ on $\partial\Omega$ for almost all $t \in (0,T)$. Hence, $F(\cdot,w(\cdot)) \in \mathbb{F}$ for any $w \in \mathbb{E}$. This proves b).

It remains to show c). Let $w, \tilde{w} \in \overline{B_{\mathbb{E}}}(0,R)$. We obtain

$$
\begin{aligned}
&\|F(\cdot,w(\cdot)) - F(\cdot,\tilde{w}(\cdot))\|_{\mathbb{F}} \\
&\leq \|\nabla\cdot((n+M)\nabla c) + u\cdot\nabla n - \nabla\cdot((\tilde{n}+M)\nabla\tilde{c}) - \tilde{u}\cdot\nabla\tilde{n}\|_{L^p(0,T;L^q_{av}(\Omega))} \\
&\quad + \|u\cdot\nabla c - \tilde{u}\cdot\nabla\tilde{c}\|_{L^p(0,T;W^{1,q}(\Omega))} \\
&\quad + \|\mathbb{P}[\kappa(u\cdot\nabla)u] - \mathbb{P}[\kappa(\tilde{u}\cdot\nabla)\tilde{u}]\|_{L^p(0,T;L^q_\sigma(\Omega))} \\
&\leq \|\nabla n\nabla c - \nabla\tilde{n}\nabla\tilde{c}\||_{L^p(0,T;L^q(\Omega))} \\
&\quad + \|(n+M)\Delta c - (\tilde{n}+M)\Delta\tilde{c}\|_{L^p(0,T;L^q(\Omega))} \\
&\quad + \|u\cdot\nabla n - \tilde{u}\cdot\nabla\tilde{n}\|_{L^p(0,T;L^q(\Omega))} + \|u\cdot\nabla c - \tilde{u}\cdot\nabla\tilde{c}\|_{L^p(0,T;W^{1,q}(\Omega))} \\
&\quad + \|\kappa(u\cdot\nabla)u - \kappa(\tilde{u}\cdot\nabla)\tilde{u}\|_{L^p(0,T;L^q(\Omega))} \\
&\leq \|\nabla(n-\tilde{n})\nabla c\|_{L^p(0,T;L^q(\Omega))} + \|\nabla\tilde{n}\nabla(c-\tilde{c})\|_{L^p(0,T;L^q(\Omega))} \\
&\quad + \|(n-\tilde{n})\Delta c\|_{L^p(0,T;L^q(\Omega))} + \|(\tilde{n}+M)\Delta(c-\tilde{c})\|_{L^p(0,T;L^q(\Omega))}
\end{aligned}
$$

$$
\begin{aligned}
&+ \|(u-\tilde u)\cdot\nabla n\|_{L^p(0,T;L^q(\Omega))} + \|\tilde u\cdot\nabla(n-\tilde n)\|_{L^p(0,T;L^q(\Omega))}\\
&+ \|(u-\tilde u)\cdot\nabla c\|_{L^p(0,T;W^{1,q}(\Omega))} + \|\tilde u\cdot\nabla(c-\tilde c)\|_{L^p(0,T;W^{1,q}(\Omega))}\\
&+ \|\kappa((u-\tilde u)\cdot\nabla)u\|_{L^p(0,T;L^q(\Omega))} + \|\kappa(\tilde u\cdot\nabla)(u-\tilde u)\|_{L^p(0,T;L^q(\Omega))}.
\end{aligned}
$$

Next, we use Hölder's inequality and Sobolev embeddings to obtain

$$
\begin{aligned}
&\|F(\cdot,w(\cdot))-F(\cdot,\tilde w(\cdot))\|_{\mathbb{F}}\\
&\le \|\nabla(n-\tilde n)\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\nabla c\|_{L^{2p}(0,T;L^{2q}(\Omega))}\\
&\quad + \|\nabla\tilde n\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\nabla(c-\tilde c)\|_{L^{2p}(0,T;L^{2q}(\Omega))}\\
&\quad + \|n-\tilde n\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\Delta c\|_{L^{2p}(0,T;L^{2q}(\Omega))}\\
&\quad + \|\tilde n\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\Delta(c-\tilde c)\|_{L^{2p}(0,T;L^{2q}(\Omega))}\\
&\quad + M\|\Delta(c-\tilde c)\|_{L^p(0,T;L^q(\Omega))} + \|u-\tilde u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\nabla n\|_{L^{2p}(0,T;L^{2q}(\Omega))}\\
&\quad + \|\tilde u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\nabla(n-\tilde n)\|_{L^{2p}(0,T;L^{2q}(\Omega))}\\
&\quad + c_0\|u-\tilde u\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\|\nabla c\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\\
&\quad + c_0\|\tilde u\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\|\nabla(c-\tilde c)\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\\
&\quad + |\kappa|\|u-\tilde u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\nabla u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\\
&\quad + |\kappa|\|\tilde u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\nabla(u-\tilde u)\|_{L^{2p}(0,T;L^{2q}(\Omega))}\\
&\le c_1\Big(\|n-\tilde n\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\|c\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\\
&\quad + \|\tilde n\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\|c-\tilde c\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\\
&\quad + \|n-\tilde n\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|c\|_{L^{2p}(0,T;W^{2,2q}(\Omega))}\\
&\quad + \|\tilde n\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|c-\tilde c\|_{L^{2p}(0,T;W^{2,2q}(\Omega))}\\
&\quad + M\|c-\tilde c\|_{L^p(0,T;W^{2,q}(\Omega))} + \|u-\tilde u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|n\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\\
&\quad + \|\tilde u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|n-\tilde n\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\\
&\quad + \|u-\tilde u\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\|c\|_{L^{2p}(0,T;W^{2,2q}(\Omega))}\\
&\quad + \|\tilde u\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\|c-\tilde c\|_{L^{2p}(0,T;W^{2,2q}(\Omega))}\\
&\quad + \|u-\tilde u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|u\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\\
&\quad + \|\tilde u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|u-\tilde u\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\Big).
\end{aligned}
$$

Similarly to the proof of b), using (3.5) and the embeddings due to the mixed derivative theorem (3.4), we obtain

$$
\begin{aligned}
&\|F(\cdot,w(\cdot))-F(\cdot,\tilde w(\cdot))\|_{\mathbb{F}}\\
&\quad\le c_2\Big(\|n-\tilde n\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|c\|_{H^{\theta,p}(0,T;H^{1+2(1-\theta),q}(\Omega))}
\end{aligned}
$$

$$
\begin{aligned}
&+ \|\tilde{n}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \|c - \tilde{c}\|_{H^{\theta,p}(0,T;H^{1+2(1-\theta),q}(\Omega))} \\
&+ \|n - \tilde{n}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \|c\|_{H^{\theta,p}(0,T;H^{1+2(1-\theta),q}(\Omega))} \\
&+ \|\tilde{n}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \|c - \tilde{c}\|_{H^{\theta,p}(0,T;H^{1+2(1-\theta),q}(\Omega))} \\
&+ M\|c - \tilde{c}\|_{\mathbb{E}_2} + \|u - \tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \|n\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \\
&+ \|\tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \|n - \tilde{n}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \\
&+ \|u - \tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \|c\|_{H^{\theta,p}(0,T;H^{1+2(1-\theta),q}(\Omega))} \\
&+ \|\tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \|c - \tilde{c}\|_{H^{\theta,p}(0,T;H^{1+2(1-\theta),q}(\Omega))} \\
&+ \|u - \tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \|u\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \\
&+ \|\tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \|u - \tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\Big) \\
\le\; & c_3\Big(2\|n - \tilde{n}\|_{\mathbb{E}_1}\|c\|_{\mathbb{E}_2} + (2\|\tilde{n}\|_{\mathbb{E}_1} + M)\|c - \tilde{c}\|_{\mathbb{E}_2} + \|u - \tilde{u}\|_{\mathbb{E}_3}\|n\|_{\mathbb{E}_1} \\
&+ \|\tilde{u}\|_{\mathbb{E}_3}\|n - \tilde{n}\|_{\mathbb{E}_1} + \|u - \tilde{u}\|_{\mathbb{E}_3}\|c\|_{\mathbb{E}_2} + \|\tilde{u}\|_{\mathbb{E}_3}\|c - \tilde{c}\|_{\mathbb{E}_2} \\
&+ \|u - \tilde{u}\|_{\mathbb{E}_3}\|u\|_{\mathbb{E}_3} + \|\tilde{u}\|_{\mathbb{E}_3}\|u - \tilde{u}\|_{\mathbb{E}_3}\Big) \\
\le\; & c_4\left(\|w\|_{\mathbb{E}} + \|\tilde{w}\|_{\mathbb{E}} + M\right)\|w - \tilde{w}\|_{\mathbb{E}} \\
\le\; & c_5(R + M)\|w - \tilde{w}\|_{\mathbb{E}}.
\end{aligned}
$$

This shows c) and therefore finishes the proof. □

After having shown the existence of a (possibly negative) periodic solutions to (PKSNS II), we are now in the position to establish nonnegative solutions to (PKSNS).

Corollary 3.1.2. *Assume that in the situation of Theorem 3.1.1, in addition, f_2 is nonnegative. Then, given $M \in (0, M_0)$, there is $r_1 \in (0, r_0]$ such that $(n + M, c + M, u)^T \in \mathbb{E}$ is a T-periodic solution to* (PKSNS) *with $f_1 \equiv 0$ and $(n + M, c + M)$ nonnegative provided $r \in (0, r_1)$.*

Proof. Recall that the mixed derivative theorem implies (3.4) for any $\theta \in (0, 1)$. Furthermore, the condition (3.2) on p and q yields the existence of a $\theta \in (0, 1)$ such that

$$\frac{1}{p} < \theta < 1 - \frac{d}{2q}.$$

This is equivalent to

$$\theta - \frac{1}{p} > 0 \quad \text{and} \quad 2(1 - \theta) - \frac{d}{q} > 0.$$

The latter furthermore implies $1 + 2(1-\theta) - \frac{d}{q} > 0$. Hence, by Sobolev embeddings we have

$$\begin{aligned} H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega)) &\hookrightarrow L^\infty(0,T;L^\infty(\Omega)), \\ H^{\theta,p}(0,T;H^{1+2(1-\theta),q}(\Omega)) &\hookrightarrow L^\infty(0,T;L^\infty(\Omega)). \end{aligned} \tag{3.6}$$

Combining the embeddings (3.4) and (3.6), we obtain

$$\mathbb{E} \hookrightarrow (L^\infty(0,T;L^\infty(\Omega)))^3$$

and therefore, there exists a $\tilde{c} > 0$ such that the T-periodic solution $(n,c,u)^T \in \overline{B_{\mathbb{E}}}(0,r)$ satisfies

$$\|(n,c,u)^T\|_{(L^\infty(0,T;L^\infty(\Omega)))^3} \leq \tilde{c}\|(n,c,u)^T\|_{\mathbb{E}} \leq \tilde{c}r.$$

Then the choice $r_1 := \min\{r_0, \frac{M}{\tilde{c}}\}$ implies that $(n+M, c+M)$ are non-negative and that $(n+M, c+M, u)^T$ is a T-periodic solution to (PKSNS) for $r \in (0, r_1)$. Hence, the proof is complete. □

Next, we consider the quasilinear case. For this purpose, for $w = (n,c,u)^T$ and $z = (z_1, z_2, z_3)^T$ we define the quasilinear operator

$$\mathcal{A}(w) := \begin{pmatrix} \nabla \cdot ((n+M+1)^m \nabla) & 0 & 0 \\ 1 & \Delta_N^1 - 1 & 0 \\ \mathbb{P}\nabla\phi & 0 & A_D \end{pmatrix},$$

i.e.,

$$\mathcal{A}(w)z := \begin{pmatrix} \nabla \cdot ((n+M+1)^m \nabla z_1) \\ z_1 + (\Delta_N^1 - 1)z_2 \\ (\mathbb{P}\nabla\phi)z_1 + A_D z_3 \end{pmatrix},$$

where $\nabla \cdot ((n+M+1)^m \nabla)$ with domain $D(\nabla \cdot ((n+M+1)^m \nabla)) = W_N^{2,q}(\Omega) \cap L_{av}^q(\Omega)$ is endowed with Neumann boundary conditions. The rest of the notation is the same as in the semilinear setting.

Then, the result on existence and uniqueness of strong time-periodic solutions to (PQKSNS) and (PQKSNS II) in the strong setting reads as follows.

Theorem 3.1.3. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with smooth boundary, $m \in \mathbb{R}$, $M > 0$, $T > 0$ and assume that* (3.2) *is satisfied. Let $f = (0, f_2, f_3)^T \in \mathbb{F}$ be T-periodic.*

Then there is $M_0 > 0$ with the property that for any $M \in (0, M_0)$ there is $r_0 = r_0(M) > 0$ such that for any $r \in (0, r_0)$ there exists $\delta = \delta(r) > 0$ such that if $\|f\|_{\mathbb{F}} < \delta$, then there exists a T-periodic solution $w = (n, c, u)^T \in \mathbb{E}$ to (PQKSNS II)*, which is unique in $\overline{B_{\mathbb{E}}}(0, r)$.*

If in addition f_2 is nonnegative, then $(n + M, c + M, u)^T \in \mathbb{E}$ is a T-periodic solution to (PQKSNS) *with $f_1 \equiv 0$ such that $(n + M, c + M)$ is nonnegative.*

Proof. Due to Proposition 2.5.6 we have to show the following properties:

a') The operator $\mathcal{A}(0)$ admits maximal periodic L^p-regularity on X_0.

b) $F(\cdot, w(\cdot)) \in \mathbb{F}$ for any $w \in \mathbb{E}$.

c) There exists a $C > 0$ such that for any $R > 0$ it is

$$\|F(\cdot, w(\cdot)) - F(\cdot, \tilde{w}(\cdot))\|_{\mathbb{F}} \leq C(R + M)\|w - \tilde{w}\|_{\mathbb{E}}$$

for all $w, \tilde{w} \in \overline{B_{\mathbb{E}}}(0, R)$.

d) There exists $R_0 > 0$ such that for each $R \in (0, R_0)$ there exists $L(R) > 0$ such that

$$\|\mathcal{A}(w(\cdot))z(\cdot) - \mathcal{A}(\tilde{w}(\cdot))z(\cdot)\|_{\mathbb{F}} \leq L(R)\|w - \tilde{w}\|_{\mathbb{E}}\|z\|_{\mathbb{E}}$$

for all $w, \tilde{w}, z \in \overline{B_{\mathbb{E}}}(0, R)$.

Assertions b) and c) are exactly the same as in the proof of Theorem 3.1.1. Therefore, they are proved in the same way.

Let $X_1 := (W^{2,q}_N(\Omega) \cap L^q_{av}(\Omega)) \times W^{3,q}_N(\Omega) \times (W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)) = D(\Delta_N) \times D(\Delta^1_N) \times D(A_D)$ and $X_\gamma := (X_0, X_1)_{1-1/p,p}$. It is

$$\mathcal{A}(0) := \begin{pmatrix} (M+1)^m \Delta_N & 0 & 0 \\ 1 & \Delta^1_N - 1 & 0 \\ \mathbb{P}\nabla\phi & 0 & A_D \end{pmatrix}.$$

Hence, by similar arguments as for assertion a) in the proof of Theorem 3.1.1, $\mathcal{A}(0)$ satisfies assertion a'). Furthermore, $\mathcal{A} : X_\gamma \to \mathcal{L}(X_1, X_0)$ is a family of closed operators. Then, it remains to prove assertion d). First, we show that the first component of $\mathcal{A}(w)$ is uniformly elliptic. Since by (3.2) the embeddings (3.4) and (3.6) are satisfied, for $\tilde{c} > 0$ we obtain

$$\|n\|_{L^\infty(0,T;L^\infty(\Omega))} \leq \tilde{c}\|n\|_{\mathbb{E}} \leq \tilde{c}R \leq M$$

for $w = (n, c, u)^T \in \overline{B_{\mathbb{E}}}(0, R)$ for some $R \in (0, R_0)$ with $R_0 := \frac{M}{\tilde{c}}$. Then, it holds $(n + M + 1) \in [1, 2M + 1]$. Consequently, the first component of $\mathcal{A}(w)$ is uniformly elliptic for all $w \in \overline{B_{\mathbb{E}}}(0, R)$.

Defining for the sake of convenience $a := M + 1$, for $w, \tilde{w}, z \in \overline{B_{\mathbb{E}}}(0, R)$ we obtain

$$\begin{aligned}
&\|\mathcal{A}(w(\cdot))z(\cdot) - \mathcal{A}(\tilde{w}(\cdot))z(\cdot)\|_{\mathbb{F}} \\
&= \|\nabla \cdot ((n + a)^m \nabla z_1) - \nabla \cdot ((\tilde{n} + a)^m \nabla z_1)\|_{L^p(0,T;L^q_{av}(\Omega))} \\
&\leq \|(m(n + a)^{m-1}\nabla n - m(\tilde{n} + a)^{m-1}\nabla \tilde{n}) \cdot \nabla z_1\|_{L^p(0,T;L^q(\Omega))} \\
&\quad + \|((n + a)^m - (\tilde{n} + a)^m)\Delta z_1\|_{L^p(0,T;L^q(\Omega))} \\
&\leq \|(m(n + a)^{m-1}(\nabla n - \nabla \tilde{n}) \cdot \nabla z_1\|_{L^p(0,T;L^q(\Omega))} \\
&\quad + \|(m(n + a)^{m-1} - m(\tilde{n} + a)^{m-1})\nabla \tilde{n} \cdot \nabla z_1\|_{L^p(0,T;L^q(\Omega))} \\
&\quad + \|((n + a)^m - (\tilde{n} + a)^m)\Delta z_1\|_{L^p(0,T;L^q(\Omega))}.
\end{aligned} \tag{3.7}$$

Using the fact that for some $c_1 > 0$ we have

$$|(n + a)^m| + |m(n + a)^{m-1}| + |m(m - 1)(n + a)^{m-2}| \leq c_1 \tag{3.8}$$

for any $w \in \overline{B_{\mathbb{E}}}(0, R)$ and Hölder's inequality in (3.7), we obtain

$$\begin{aligned}
&\|\mathcal{A}(w(\cdot))z(\cdot) - \mathcal{A}(\tilde{w}(\cdot))z(\cdot)\|_{\mathbb{F}} \\
&\leq c_1\Big(\|\nabla(n - \tilde{n})\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\nabla z_1\|_{L^{2p}(0,T;L^{2q}(\Omega))} \\
&\quad + \|(n - \tilde{n})\nabla \tilde{n} \cdot \nabla z_1\|_{L^p(0,T;L^q(\Omega))} + \|(n - \tilde{n})\Delta z_1\|_{L^p(0,T;L^q(\Omega))}\Big) \\
&\leq c_1\Big(\|n - \tilde{n}\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\|z_1\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} \\
&\quad + \|n - \tilde{n}\|_{L^\infty(0,T;L^\infty(\Omega))}\|\nabla \tilde{n}\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\nabla z_1\|_{L^{2p}(0,T;L^{2q}(\Omega))} \\
&\quad + \|n - \tilde{n}\|_{L^\infty(0,T;L^\infty(\Omega))}\|\Delta z_1\|_{L^p(0,T;L^q(\Omega))}\Big).
\end{aligned}$$

Recall that due to (3.2) we have the embeddings (3.4), (3.5), and (3.6). Hence, we obtain

$$
\begin{aligned}
&\|\mathcal{A}(w(\cdot))z(\cdot) - \mathcal{A}(\tilde{w}(\cdot))z(\cdot)\|_{\mathbb{F}} \\
&\le c_1\|n-\tilde{n}\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\|z_1\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} \\
&\quad + c_2\|n-\tilde{n}\|_{\mathbb{E}_1}\|\tilde{n}\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\|z_1\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} \\
&\quad + c_2\|n-\tilde{n}\|_{\mathbb{E}_1}\|z_1\|_{L^{p}(0,T;W^{2,q}(\Omega))} \\
&\le c_3\Big(\|n-\tilde{n}\|_{\mathbb{E}_1}\|z_1\|_{\mathbb{E}_1} + \|n-\tilde{n}\|_{\mathbb{E}_1}\|\tilde{n}\|_{\mathbb{E}_1}\|z_1\|_{\mathbb{E}_1} + \|n-\tilde{n}\|_{\mathbb{E}_1}\|z_1\|_{\mathbb{E}_1}\Big) \\
&\le c_3(R+2)\|w-\tilde{w}\|_{\mathbb{E}}\|z\|_{\mathbb{E}}.
\end{aligned}
$$

This proves d). Hence, we proved a'), b), c), and d). This yields the first part of the theorem.

The claim concerning the nonnegativity of the solution can be proved in the same way as in Corollary 3.1.2. □

3.1.2 Weak Setting

In this subsection we study strong periodic solutions to the chemotaxis–Navier–Stokes system in the $W^{-1,q}(\Omega) \times L^q(\Omega) \times L^q_\sigma(\Omega)$-setting. We proceed as in the strong setting. This time, the ground space X_0 is given by

$$
X_0 := \Big(W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega)\Big)' \times L^q(\Omega) \times L^q_\sigma(\Omega). \tag{3.9}
$$

We consider again (3.1), where in the weak setting, the operator $\mathcal{A}$ on X_0 and, for $w = (n, c, u)^T$, the mapping F are given by

$$
\mathcal{A} := \begin{pmatrix} \Delta_{N,w} & 0 & 0 \\ 1 & \Delta_N - 1 & 0 \\ \mathbb{P}\nabla\phi & 0 & A_D \end{pmatrix}, \tag{3.10}
$$

$$
F(t,w) := \begin{pmatrix} -\nabla\cdot((n+M)\nabla c) - u\cdot\nabla n \\ -u\cdot\nabla c + f_2(t) \\ \mathbb{P}[\kappa(u\cdot\nabla)u] + \mathbb{P}[M\nabla\phi] + \mathbb{P}f_3(t) \end{pmatrix}, \tag{3.11}
$$

where $\Delta_{N,w}$ with domain $D(\Delta_{N,w}) = W^{1,q}(\Omega) \cap L^q_{av}(\Omega)$ denotes the Neumann-Laplacian on $(W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'$ and Δ_N with domain $D(\Delta_N) = W^{2,q}_N(\Omega)$ denotes the Neumann-Laplacian on $L^q(\Omega)$. Moreover, as before A_D with domain $D(A_D) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$ denotes the Stokes operator on $L^q_\sigma(\Omega)$.

Then, the data and solution spaces in the weak setting are defined as

(3.12)
$$\begin{aligned}
\mathbb{F} &:= L^p(0,T;X_0)\\
\mathbb{E}_1 &:= L^p(0,T;W^{1,q}(\Omega) \cap L^q_{av}(\Omega)) \cap W^{1,p}(0,T;(W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'),\\
\mathbb{E}_2 &:= L^p(0,T;W^{2,q}_N(\Omega)) \cap W^{1,p}(0,T;L^q(\Omega)),\\
\mathbb{E}_3 &:= L^p(0,T;W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)) \cap W^{1,p}(0,T;L^q_\sigma(\Omega)),\\
\mathbb{E} &:= \mathbb{E}_1 \times \mathbb{E}_2 \times \mathbb{E}_3.
\end{aligned}$$

Provided that

$$p, q \in (1,\infty) \quad \text{such that} \quad q > \frac{d}{2} \quad \text{and} \quad \frac{1}{p} + \frac{d}{2q} < 1 \tag{3.13}$$

as well as $q' \in (1,\infty)$ with $\frac{1}{q} + \frac{1}{q'} = 1$, our result on existence of strong time-periodic solutions to (PKSNS II) in the weak setting reads as follows.

Theorem 3.1.4. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded domain with smooth boundary, $M > 0$, $T > 0$, and assume that* (3.13) *is satisfied. Moreover, let X_0, $\mathbb{E}$, and $\mathbb{F}$ be defined as in* (3.9) *and* (3.12) *and let $f = (0, f_2, f_3)^T \in \mathbb{F}$ be T-periodic.*

Then there are $r_0 > 0$ and $M_0 > 0$ such that for any $r \in (0, r_0)$ there exists $\delta = \delta(r) > 0$ such that if $\|f\|_{\mathbb{F}} < \delta$ and $M < M_0$, then there exists a T-periodic solution $w = (n, c, u)^T \in \mathbb{E}$ to (PKSNS II)*, which is unique in $\overline{B_{\mathbb{E}}}(0, r)$.*

Proof. We proceed as in the proof of Theorem 3.1.1. Hence, we have to verify the assertions a), b), and c) stated there, but this time for the notation of the weak setting, namely, for the spaces X_0, $\mathbb{E}$, and $\mathbb{F}$ defined in (3.9) and (3.12) and the operator $\mathcal{A}$ and right-hand side F defined in (3.10).

Due to Proposition 2.4.1 and 2.4.3, the operators $\Delta_{N,w}$ and $(\Delta_N - 1)$ admit maximal L^p-regularity on $(W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'$ and $L^q(\Omega)$, respectively. Furthermore both are invertible operators. Since the same holds true for the Stokes operator A_D on $L^q_\sigma(\Omega)$, see Proposition 2.4.8, due to the triangular structure of $\mathcal{A}$ we see as in the strong setting that $\mathcal{A}$ admits maximal L^p-regularity on X_0. Hence, Proposition 2.2.1 proves assertion a).

Next, we show that $F(\cdot, w(\cdot)) \in \mathbb{F}$ is satisfied for all $w \in \mathbb{E}$. Using that u is divergence-free, ∇ is a bounded operator from $L^q(\Omega) = (L^{q'}(\Omega))'$ to $(W^{1,q'}(\Omega))'$, and $(W^{1,q'}(\Omega))' \subset (W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'$, we obtain for $w \in \mathbb{E}$ the estimate

$$
\begin{aligned}
&\|F(\cdot, w(\cdot))\|_{\mathbb{F}} \\
&\le \| -\nabla \cdot ((n+M)\nabla c) - \nabla \cdot (un)\|_{L^p(0,T;(W^{1,q'}(\Omega)\cap L^{q'}_{av}(\Omega))')} \\
&\quad + \| -u\cdot\nabla c + f_2(t)\|_{L^p(0,T;L^q(\Omega))} \\
&\quad + \|\mathbb{P}[\kappa(u\cdot\nabla)u] + \mathbb{P}[M\nabla\phi] + \mathbb{P}f_3(t)\|_{L^p(0,T;L^q_\sigma(\Omega))} \\
&\le c_1\left(\|(n+M)\nabla c\|_{L^p(0,T;L^q(\Omega))} + \|un\|_{L^p(0,T;L^q(\Omega))}\right) + \|u\cdot\nabla c\|_{L^p(0,T;L^q(\Omega))} \\
&\quad + \|\kappa(u\cdot\nabla)u\|_{L^p(0,T;L^q(\Omega))} + \|M\nabla\phi\|_{L^p(0,T;L^q(\Omega))} + \|f\|_{\mathbb{F}}.
\end{aligned}
$$

Then, Hölder's inequality yields

$$
\begin{aligned}
&\|F(\cdot, w(\cdot))\|_{\mathbb{F}} \\
&\le c_1\Big(\|n\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\nabla c\|_{L^{2p}(0,T;L^{2q}(\Omega))} + M\|\nabla c\|_{L^p(0,T;L^q(\Omega))} \\
&\quad + \|u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|n\|_{L^{2p}(0,T;L^{2q}(\Omega))}\Big) \\
&\quad + \|u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\nabla c\|_{L^{2p}(0,T;L^{2q}(\Omega))} \\
&\quad + |\kappa|\|u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|\nabla u\|_{L^{2p}(0,T;L^{2q}(\Omega))} + M\|\nabla\phi\|_{L^p(0,T;L^q(\Omega))} + \|f\|_{\mathbb{F}} \\
&\le c_2\Big(\|n\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|c\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} + M\|c\|_{L^p(0,T;W^{1,q}(\Omega))} \\
&\quad + \|u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|n\|_{L^{2p}(0,T;L^{2q}(\Omega))} + \|u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|c\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} \\
&\quad + \|u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|u\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\Big) + M\|\nabla\phi\|_{L^p(0,T;L^q(\Omega))} + \|f\|_{\mathbb{F}}.
\end{aligned}
\tag{3.14}
$$

Following the idea of the proof in the strong setting, we want to use the mixed derivative theorem and Sobolev embeddings. Due to the Proposition 2.5.1 we have

$$
\begin{aligned}
&\mathbb{E}_1 \hookrightarrow H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega)),\ \mathbb{E}_2 \hookrightarrow H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega)), \\
&\mathbb{E}_3 \hookrightarrow H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))
\end{aligned}
\tag{3.15}
$$

for any $\theta \in (0,1)$. Furthermore, due to (3.13) we deduce the existence of $\theta \in (0,1)$ satisfying

$$\frac{1}{2p} < \theta < \frac{1}{2} - \frac{d}{4q}.$$

This inequality is equivalent to

$$\theta - \frac{1}{p} > -\frac{1}{2p}, \quad 2(1-\theta) - \frac{d}{q} > 1 - \frac{d}{2q}.$$

Hence, by Sobolev embeddings we obtain

$$\begin{aligned}
&H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega)) \hookrightarrow L^{2p}(0,T;W^{1,2q}(\Omega)) \hookrightarrow L^{2p}(0,T;L^{2q}(\Omega)),\\
&H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega)) \hookrightarrow L^{2p}(0,T;L^{2q}(\Omega)).
\end{aligned} \tag{3.16}$$

Using the Sobolev embeddings (3.16) and the embeddings due to the mixed derivative theorem (3.15) in (3.14), we obtain

$$\begin{aligned}
&\|F(\cdot,w(\cdot))\|_{\mathbb{F}}\\
&\quad\le c_3\Big(\|n\|_{H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega))}\|c\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} + M\|c\|_{\mathbb{E}_2}\\
&\qquad + \|u\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|n\|_{H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega))}\\
&\qquad + \|u\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|c\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\\
&\qquad + \|u\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|u\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\Big)\\
&\qquad + M\|\nabla\phi\|_{L^p(0,T;L^q(\Omega))} + \|f\|_{\mathbb{F}}\\
&\quad\le c_4\Big(\|n\|_{\mathbb{E}_1}\|c\|_{\mathbb{E}_2} + \|u\|_{\mathbb{E}_3}\|n\|_{\mathbb{E}_1} + \|u\|_{\mathbb{E}_3}\|c\|_{\mathbb{E}_2} + \|u\|_{\mathbb{E}_3}\|u\|_{\mathbb{E}_3} + M\|c\|_{\mathbb{E}_2}\Big)\\
&\qquad + M\|\nabla\phi\|_{L^p(0,T;L^q(\Omega))} + \|f\|_{\mathbb{F}}\\
&\quad\le c_5(\|w\|_{\mathbb{E}}^2 + M\|w\|_{\mathbb{E}}) + M\|\nabla\phi\|_{L^p(0,T;L^q(\Omega))} + \|f\|_{\mathbb{F}}.
\end{aligned}$$

Hence, in view of $(W^{1,q'}(\Omega))' \subset (W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'$, we conclude that $F(\cdot,w(\cdot)) \in \mathbb{F}$. This proves b).

It remains to verify assertion c). In order to do so, let w, $\tilde{w} \in \overline{B_{\mathbb{E}}}(0,R)$. By $\nabla\cdot u = 0$, the boundedness of ∇ from $L^q(\Omega) = (L^{q'}(\Omega))'$ to $(W^{1,q'}(\Omega))'$, and $(W^{1,q'}(\Omega))' \subset (W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'$ we obtain

$$
\begin{aligned}
&\|F(\cdot, w(\cdot)) - F(\cdot, \tilde{w}(\cdot))\|_{\mathbb{F}} \\
&\quad \le \|\nabla \cdot ((n+M)\nabla c - (\tilde{n}+M)\nabla \tilde{c})\|_{L^p(0,T;(W^{1,q'}(\Omega))')} \\
&\qquad + \|\nabla \cdot (un - \tilde{u}\tilde{n})\|_{L^p(0,T;(W^{1,q'}(\Omega))')} \\
&\qquad + \|u \cdot \nabla c - \tilde{u} \cdot \nabla \tilde{c}\|_{L^p(0,T;L^q(\Omega))} + \|\kappa(u \cdot \nabla)u - \kappa(\tilde{u} \cdot \nabla)\tilde{u}\|_{L^p(0,T;L^q(\Omega))} \\
&\quad \le c_1 \Big\|(n+M)\nabla c - (\tilde{n}+M)\nabla \tilde{c}\|_{L^p(0,T;L^q(\Omega))} + \|un - \tilde{u}\tilde{n}\|_{L^p(0,T;L^q(\Omega))}\Big) \\
&\qquad + \|u \cdot \nabla c - \tilde{u} \cdot \nabla \tilde{c}\|_{L^p(0,T;L^q(\Omega))} + \|\kappa(u \cdot \nabla)u - \kappa(\tilde{u} \cdot \nabla)\tilde{u}\|_{L^p(0,T;L^q(\Omega))} \\
&\quad \le c_1 \Big(\|(n-\tilde{n})\nabla c\|_{L^p(0,T;L^q(\Omega))} + \|(\tilde{n}+M)\nabla(c-\tilde{c})\|_{L^p(0,T;L^q(\Omega))} \\
&\qquad + \|(u-\tilde{u})n\|_{L^p(0,T;L^q(\Omega))} + \|\tilde{u}(n-\tilde{n})\|_{L^p(0,T;L^q(\Omega))}\Big) \\
&\qquad + \|(u-\tilde{u}) \cdot \nabla c\|_{L^p(0,T;L^q(\Omega))} + \|\tilde{u} \cdot \nabla(c-\tilde{c})\|_{L^p(0,T;L^q(\Omega))} \\
&\qquad + \|\kappa((u-\tilde{u}) \cdot \nabla)u\|_{L^p(0,T;L^q(\Omega))} + \|\kappa(\tilde{u} \cdot \nabla)(u-\tilde{u})\|_{L^p(0,T;L^q(\Omega))}.
\end{aligned}
$$

Next, we employ Hölder's inequality to obtain

$$
\begin{aligned}
&\|F(\cdot, w(\cdot)) - F(\cdot, \tilde{w}(\cdot))\|_{\mathbb{F}} \\
&\quad \le c_2 \Big(\|n-\tilde{n}\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|c\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} \\
&\qquad + \|\tilde{n}\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|c-\tilde{c}\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} + M\|c-\tilde{c}\|_{L^p(0,T;W^{1,q}(\Omega))} \\
&\qquad + \|u-\tilde{u}\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|n\|_{L^{2p}(0,T;L^{2q}(\Omega))} \\
&\qquad + \|\tilde{u}\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|n-\tilde{n}\|_{L^{2p}(0,T;L^{2q}(\Omega))} \\
&\qquad + \|u-\tilde{u}\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|c\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} \\
&\qquad + \|\tilde{u}\|_{L^{2}(0,T;L^{2q}(\Omega))}\|c-\tilde{c}\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} \\
&\qquad + \|u-\tilde{u}\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|u\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} \\
&\qquad + \|\tilde{u}\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|u-\tilde{u}\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}\Big).
\end{aligned}
$$

Similarly to the proof of b), using the Sobolev embeddings (3.16) and the embeddings due to the mixed derivative theorem (3.15), we obtain

$$
\begin{aligned}
&\|F(\cdot, w(\cdot)) - F(\cdot, \tilde{w}(\cdot))\|_{\mathbb{F}} \\
&\quad \le c_3 \Big(\|n-\tilde{n}\|_{H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega))}\|c\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \\
&\qquad + \|\tilde{n}\|_{H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega))}\|c-\tilde{c}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} + M\|c-\tilde{c}\|_{\mathbb{E}_2} \\
&\qquad + \|u-\tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|n\|_{H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega))}
\end{aligned}
$$

$$
\begin{aligned}
&+ \|\tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|n-\tilde{n}\|_{H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega))}\\
&+ \|u-\tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|c\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\\
&+ \|\tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|c-\tilde{c}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\\
&+ \|u-\tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|u\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\\
&+ \|\tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|u-\tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\Big)\\
\le{}& c_4\Big(\|n-\tilde{n}\|_{\mathbb{E}_1}\|c\|_{\mathbb{E}_2}+\|\tilde{n}\|_{\mathbb{E}_1}\|c-\tilde{c}\|_{\mathbb{E}_2}+M\|c-\tilde{c}\|_{\mathbb{E}_2}\\
&+\|u-\tilde{u}\|_{\mathbb{E}_3}\|n\|_{\mathbb{E}_1}+\|\tilde{u}\|_{\mathbb{E}_3}\|n-\tilde{n}\|_{\mathbb{E}_1}+\|u-\tilde{u}\|_{\mathbb{E}_3}\|c\|_{\mathbb{E}_2}\\
&+\|\tilde{u}\|_{\mathbb{E}_3}\|c-\tilde{c}\|_{\mathbb{E}_2}+\|u-\tilde{u}\|_{\mathbb{E}_3}\|u\|_{\mathbb{E}_3}+\|\tilde{u}\|_{\mathbb{E}_3}\|u-\tilde{u}\|_{\mathbb{E}_3}\Big)\\
\le{}& c_5\left(\|w\|_{\mathbb{E}}+\|\tilde{w}\|_{\mathbb{E}}+M\right)\|w-\tilde{w}\|_{\mathbb{E}}\\
\le{}& c_6(R+M)\|w-\tilde{w}\|_{\mathbb{E}}.
\end{aligned}
$$

This proves c). Since the assertion of the theorem follows from a), b), and c) in the exact same way as in the proof of Theorem 3.1.1, the proof is complete. □

As in the strong setting, after having shown the existence of a (possibly negative) periodic solutions to (PKSNS II), we are now in the position to establish nonnegative solutions to (PKSNS).

Corollary 3.1.5. *Assume that in the situation of Theorem 3.1.4, in addition, f_2 is nonnegative and*

$$
p>2,\quad q>d,\quad \text{and}\quad \frac{1}{p}+\frac{d}{2q}<\frac{1}{2} \tag{3.17}
$$

is satisfied. Then, given $M\in(0,M_0)$, there is $r_1\in(0,r_0]$ such that $(n+M,c+M,u)^T\in\mathbb{E}$ is a T-periodic solution with $(n+M,c+M)$ nonnegative to (PKSNS) *with $f_1\equiv 0$ provided $r\in(0,r_1)$.*

Proof. Recall that the mixed derivative theorem implies (3.15) for any $\theta\in(0,1)$. Furthermore, the condition (3.17) on p and q yields the existence of a $\theta\in(0,1)$ such that

$$
\frac{1}{p}<\theta<\frac{1}{2}-\frac{d}{2q}.
$$

This is equivalent to

$$\theta - \frac{1}{p} > 0 \quad \text{and} \quad -1 + 2(1-\theta) - \frac{d}{q} > 0.$$

The latter furthermore implies $2(1-\theta) - \frac{d}{q} > 0$. Hence, by Sobolev embeddings it holds

$$\begin{aligned} &H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega)) \hookrightarrow L^\infty(0,T;L^\infty(\Omega)),\\ &H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega)) \hookrightarrow L^\infty(0,T;L^\infty(\Omega)). \end{aligned} \tag{3.18}$$

Combining the embeddings (3.15) and (3.18), we obtain

$$\mathbb{E} \hookrightarrow (L^\infty(0,T;L^\infty(\Omega)))^3$$

and therefore, there exists a $\tilde{c} > 0$ such that the T-periodic $(n,c,u)^T \in \overline{B_{\mathbb{E}}}(0,r)$ satisfies

$$\|(n,c,u)^T\|_{(L^\infty(0,T;L^\infty(\Omega)))^3} \le \tilde{c}\|(n,c,u)^T\|_{\mathbb{E}} \le \tilde{c}r.$$

Then the choice $r_1 := \min\{r_0, \frac{M}{\tilde{c}}\}$ implies that $(n+M, c+M)$ are non-negative and that $(n+M, c+M, u)^T$ is a T-periodic solution to (PKSNS) for $r \in (0, r_1)$. Hence, the proof is complete. □

Next, we want to consider the quasilinear case. For this purpose, for $w = (n,c,u)^T$ and $z = (z_1, z_2, z_3)^T$ we define the quasilinear operator

$$\mathcal{A}(w) := \begin{pmatrix} \nabla \cdot ((n+M+1)^m \nabla) & 0 & 0 \\ 1 & \Delta_N - 1 & 0 \\ \mathbb{P}\nabla\phi & 0 & A_D \end{pmatrix},$$

i.e.,

$$\mathcal{A}(w)z := \begin{pmatrix} \nabla \cdot ((n+M+1)^m \nabla z_1) \\ z_1 + (\Delta_N - 1) z_2 \\ (\mathbb{P}\nabla\phi) z_1 + A_D z_3 \end{pmatrix}, \tag{3.19}$$

where $\nabla \cdot ((n+M+1)^m \nabla)$ with domain $D(\nabla \cdot ((n+M+1)^m \nabla)) = W^{1,q}(\Omega) \cap L^q_{av}(\Omega)$ is endowed with Neumann boundary conditions. The rest of the notation is the same as in the semilinear setting.

Then, the result on existence and uniqueness of strong time-periodic solutions to (PQKSNS) and (PQKSNS II) in the weak setting reads as follows.

Theorem 3.1.6. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with smooth boundary, $m \in \mathbb{R}$, $M > 0$, $T > 0$ and assume that* (3.17) *is satisfied. Let $f = (0, f_2, f_3)^T \in \mathbb{F}$ be T-periodic.*

Then there is $M_0 > 0$ with the property that for any $M \in (0, M_0)$ there is $r_0 = r_0(M) > 0$ such that for any $r \in (0, r_0)$ there exists $\delta = \delta(r) > 0$ such that if $\|f\|_{\mathbb{F}} < \delta$, then there exists a T-periodic solution $w = (n, c, u)^T \in \mathbb{E}$ to (PQKSNS II)*, which is unique in $\overline{B_{\mathbb{E}}}(0, r)$.*

If in addition f_2 is nonnegative, then $(n+M, c+M, u)^T \in \mathbb{E}$ is a T-periodic solution to (PQKSNS) *with $f_1 \equiv 0$ such that $(n+M, c+M)$ is nonnegative.*

Proof. We proceed as in the proof of Theorem 3.1.3. Hence, we have to verify the assertions a') and d) stated there, but this time for the notation of the weak setting, namely, for the spaces X_0, $\mathbb{E}$, and $\mathbb{F}$ defined in (3.9) and (3.12) and the operator $\mathcal{A}(w)$ defined in (3.19). The assertion b) and c) remain the same as before, hence they are proved in the same way.

Let $X_1 := W^{1,q}(\Omega) \cap L^q_{av}(\Omega) \times W^{2,q}(\Omega) \times W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) = D(\Delta_{N,w}) \times D(\Delta_N) \times D(A_D)$ and $X_\gamma := (X_0, X_1)_{1-1/p,p}$. It is

$$\mathcal{A}(0) := \begin{pmatrix} (M+1)^m \Delta_{N,w} & 0 & 0 \\ 1 & \Delta_N - 1 & 0 \\ \mathbb{P}\nabla\phi & 0 & A_D \end{pmatrix}.$$

Hence, by similar arguments as for assertion a) in the proof of Theorem 3.1.4, $\mathcal{A}(0)$ satisfies assertion a'). Furthermore, $\mathcal{A} : X_\gamma \to \mathcal{L}(X_1, X_0)$ is a family of closed operators. Then, it remains to prove assertion d). First, we show that the first component of $\mathcal{A}(w)$ is uniformly elliptic. Since by (3.17) the embeddings (3.15) and (3.18) are satisfied, for $\tilde{c} > 0$ we obtain

$$\|n\|_{L^\infty(0,T;L^\infty(\Omega))} \leq \tilde{c}\|n\|_{\mathbb{E}} \leq \tilde{c}R \leq M$$

for $w = (n, c, u)^T \in \overline{B_{\mathbb{E}}}(0, R)$ for some $R \in (0, R_0)$ with $R_0 := \frac{M}{\tilde{c}}$. Then, it is $(n + M + 1) \in [1, 2M + 1]$. Consequently the first component of $\mathcal{A}(w)$ is uniformly elliptic for all $w \in \overline{B_{\mathbb{E}}}(0, R)$.

Using that ∇ is a bounded operator from $L^q(\Omega) = (L^{q'}(\Omega))'$ to $(W^{1,q'}(\Omega))'$ and $(W^{1,q'}(\Omega))' \subset (W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'$, for $w, \tilde{w}, z \in \overline{B_{\mathbb{E}}}(0, R)$ we obtain

$$\begin{aligned}
&\|\mathcal{A}(w(\cdot))z(\cdot) - \mathcal{A}(\tilde{w}(\cdot))z(\cdot)\|_{\mathbb{F}} \\
&= \|\nabla \cdot ((n + M + 1)^m \nabla z_1 - (\tilde{n} + M + 1)^m \nabla z_1)\|_{L^p(0,T;(W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))')} \\
&\le c_1 \|(n + M + 1)^m \nabla z_1 - (\tilde{n} + M + 1)^m \nabla z_1\|_{L^p(0,T;L^q(\Omega))}.
\end{aligned}$$

Finally, we use (3.8) and the embbedings (3.15) and (3.18) to obtain

$$\begin{aligned}
\|\mathcal{A}(w(\cdot))z(\cdot) - \mathcal{A}(\tilde{w}(\cdot))z(\cdot)\|_{\mathbb{F}} &\le c_2 \|(n - \tilde{n}) \nabla z_1\|_{L^p(0,T;L^q(\Omega))} \\
&\le c_2 \|n - \tilde{n}\|_{L^\infty(0,T;L^\infty(\Omega))} \|\nabla z_1\|_{L^p(0,T;L^q(\Omega))} \\
&\le c_3 \|n - \tilde{n}\|_{\mathbb{E}_1} \|z_1\|_{L^p(0,T;W^{1,q}(\Omega))} \\
&\le c_4 \|w - \tilde{w}\|_{\mathbb{E}} \|z\|_{\mathbb{E}}
\end{aligned}$$

This proves d). Hence, we proved a'), b), c), and d). This yields the first part of the theorem.

The claim concerning the nonnegativity of the solution can be proved in the same way as in Corollary 3.1.5. □

3.2 The Initial Value Problem

In this section, we show local well-posedness for the coupled Keller–Segel–Navier–Stokes initial value problem

$$\text{(IVKSNS)} \quad \begin{cases}
\partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) & \text{in } (0, \infty) \times \Omega, \\
\partial_t c + u \cdot \nabla c = \Delta c - c + n & \text{in } (0, \infty) \times \Omega, \\
\partial_t u - \Delta u - \nabla P = \kappa (u \cdot \nabla) u + n \nabla \phi & \text{in } (0, \infty) \times \Omega, \\
\nabla \cdot u = 0 & \text{in } (0, \infty) \times \Omega, \\
\dfrac{\partial n}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = 0 & \text{on } (0, \infty) \times \partial\Omega, \\
u = 0 & \text{on } (0, \infty) \times \partial\Omega, \\
(n, c, u)(x, 0) = (n_0, c_0, u_0)(x) & \text{in } \Omega.
\end{cases}$$

by applying the abstract theory introduced in Subsection 2.5.2.

As for the time-periodic case, the abstract theory will be applied in two different settings, the strong and the weak setting.

3.2.1 Strong Setting

Let $p, q \in (1, \infty)$ and $\mu \in (1/p, 1]$. In the strong setting, as underlying ground space we choose the space $X_0 = L^q(\Omega) \times W^{1,q}(\Omega) \times L^q_\sigma(\Omega)$. Furthermore, we define the time-weighted solution and data spaces by

$$
\begin{aligned}
&\mathbb{E}_{1,\mu} := L^p_\mu(0,T; W^{2,q}_N(\Omega)) \cap H^{1,p}_\mu(0,T; L^q(\Omega)),\\
&\mathbb{E}_{2,\mu} := L^p_\mu(0,T; W^{3,q}_N(\Omega)) \cap H^{1,p}_\mu(0,T; W^{1,q}(\Omega)),\\
&\mathbb{E}_{3,\mu} := L^p_\mu(0,T; W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)) \cap H^{1,p}_\mu(0,T; L^q_\sigma(\Omega)),
\end{aligned}
\tag{3.20}
$$

as well as

$$
\mathbb{F}_\mu := L^p_\mu(0,T; X_0) \quad \text{and} \quad \mathbb{E}_\mu := \mathbb{E}_{1,\mu} \times \mathbb{E}_{2,\mu} \times \mathbb{E}_{3,\mu}. \tag{3.21}
$$

By real interpolation we obtain for the trace space, see, e.g., [5] and [72],

$$
X_{\gamma,\mu} = B^{2\mu-2/p}_{qp,N}(\Omega) \times B^{2\mu-2/p+1}_{qp,N}(\Omega) \times B^{2\mu-2/p}_{qp,0}(\Omega) \cap L^q_\sigma(\Omega). \tag{3.22}
$$

In order to apply the abstract theory of Subsection 2.5.2 to the coupled chemotaxis-Navier–Stokes system (IVKSNS), on X_0 we define the operator $\mathcal{A}$ and the mapping F as

$$
\mathcal{A} := \begin{pmatrix} \Delta_N & 0 & 0 \\ 1 & \Delta^1_N - 1 & 0 \\ \mathbb{P}\nabla\phi & 0 & A_D \end{pmatrix}, \quad F(w) := \begin{pmatrix} -\nabla \cdot (n\nabla c) - u \cdot \nabla n \\ -u \cdot \nabla c \\ \mathbb{P}\kappa(u \cdot \nabla)u \end{pmatrix}
$$

for $w = (n, c, u)^T$. Here Δ_N with domain $D(\Delta_N) = W^{2,q}_N(\Omega)$ denotes the Neumann-Laplacian on $L^q(\Omega)$ and Δ^1_N with domain $D(\Delta^1_N) = W^{3,q}_N(\Omega)$ denotes the Neumann-Laplacian on $W^{1,q}(\Omega)$. Furthermore, as before A_D with domain $D(A_D) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$ denotes the Stokes operator on $L^q_\sigma(\Omega)$.

Then, using this notation and the Helmholtz projection $\mathbb{P}$, we may rewrite (IVKSNS) as

$$\begin{cases} \partial_t w(t) - \mathcal{A}w(t) = F(w(t)) & t \in (0,\infty), \\ \qquad\qquad w(0) = w_0 \end{cases} \tag{3.23}$$

with $w_0 = (n_0, c_0, u_0)^T$.

Assuming that either

$$\begin{aligned} &p, q \in (1,\infty),\ \mu \in (1/p, 1] \quad \text{such that} \\ &\frac{d}{2} < q \leq 2 \quad \text{and} \quad \frac{2}{p} + \frac{d}{2q} < 2\mu - 1 \end{aligned} \tag{3.24}$$

or

$$p, q \in (1,\infty),\ \mu \in (1/p, 1] \quad \text{such that} \quad \frac{2}{p} + \frac{d}{q} < 2\mu - 1 \tag{3.25}$$

holds, our result on the existence of local strong solutions to (IVKSNS) reads as follows.

Theorem 3.2.1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with smooth boundary and assume that either* (3.24) *or* (3.25) *is satisfied. Moreover, let X_0, $\mathbb{E}_\mu$, $\mathbb{F}_\mu$, and $X_{\gamma,\mu}$ be defined as in* (3.20), (3.21), *and* (3.22). *Let $w_0 = (n_0, c_0, u_0)^T \in X_{\gamma,\mu}$.*

Then there exists $T = T(w_0) > 0$ such that (IVKSNS) *admits a unique solution $w = (n, c, u)^T \in \mathbb{E}_\mu$. In addition, for each $\delta \in (0, T)$ we have $w \in C([\delta, T]; X_\gamma)$, i.e., the solution regularizes instantaneously.*

Proof. In order to prove the theorem, by Proposition 2.5.3 we need to show the following properties:

a) The operator $\mathcal{A}$ admits maximal L^p-regularity on X_0.

b) $F(w) \in X_0$ for any $w \in X_{\gamma,\mu}$.

c) There exists a $C > 0$ such that for any $R > 0$ it holds

$$\|F(w) - F(\tilde{w})\|_{X_0} \leq CR\|w - \tilde{w}\|_{X_{\gamma,\mu}}$$

for all $w, \tilde{w} \in \overline{B_{X_{\gamma,\mu}}}(0, R)$.

First we show assertion a). Since Δ_N on $L^q(\Omega)$, Δ_N^1 on $W^{1,q}(\Omega)$, and A_D on $L^q_\sigma(\Omega)$ admit maximal L^p-regularity (see Section 2.4), it follows by the triangular structure that $\mathcal{A}$ admits maximal L^p-regularity on X_0. This proves a).

Next, we show that $F(w) \in X_0$ is satisfied for $w \in X_{\gamma,\mu}$. In order to do so, we estimate

$$\begin{aligned}
&\|F(w)\|_{X_0}\\
&\leq \|\nabla\cdot(n\nabla c) + u\cdot\nabla n\|_{L^q(\Omega)} + \|u\cdot\nabla c\|_{W^{1,q}(\Omega)} + \|\mathbb{P}[\kappa(u\cdot\nabla)u]\|_{L^q_\sigma(\Omega)}\\
&\leq \|\nabla n\nabla c\|_{L^q(\Omega)} + \|n\Delta c\|_{L^q(\Omega)} + \|u\cdot\nabla n\|_{L^q(\Omega)} + \|u\cdot\nabla c\|_{W^{1,q}(\Omega)}\\
&\quad + \|\kappa(u\cdot\nabla)u\|_{L^q(\Omega)}.
\end{aligned}$$

By Hölder's inequality and Sobolev embeddings we obtain

$$\begin{aligned}
&\|F(w)\|_{X_0}\\
&\leq \|\nabla n\|_{L^{2q}(\Omega)}\|\nabla c\|_{L^{2q}(\Omega)} + \|n\|_{L^{2q}(\Omega)}\|\Delta c\|_{L^{2q}(\Omega)} + \|u\|_{L^{2q}(\Omega)}\|\nabla n\|_{L^{2q}(\Omega)}\\
&\quad + c_0\|u\|_{W^{1,2q}(\Omega)}\|\nabla c\|_{W^{1,2q}(\Omega)} + |\kappa|\|u\|_{L^{2q}(\Omega)}\|\nabla u\|_{L^{2q}(\Omega)}\\
&\leq c_1\Big(\|n\|_{W^{1,2q}(\Omega)}\|c\|_{W^{1,2q}(\Omega)} + \|n\|_{L^{2q}(\Omega)}\|c\|_{W^{2,2q}(\Omega)} + \|u\|_{L^{2q}(\Omega)}\|n\|_{W^{1,2q}(\Omega)}\\
&\quad + \|u\|_{W^{1,2q}(\Omega)}\|c\|_{W^{2,2q}(\Omega)} + \|u\|_{L^{2q}(\Omega)}\|u\|_{W^{1,2q}(\Omega)}\Big).
\end{aligned} \tag{3.26}$$

Assuming (3.24), we have the following Sobolev embeddings

$$\begin{aligned}
H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega) &\hookrightarrow W^{1,2q}(\Omega),\\
H^{2\mu-\frac{2}{p}+1-\varepsilon,q}(\Omega) &\hookrightarrow W^{2,2q}(\Omega).
\end{aligned} \tag{3.27}$$

Hence, using these embeddings, we obtain

$$\begin{aligned}
&\|F(w)\|_{X_0}\\
&\leq c_2\Big(\|n\|_{H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega)}\|c\|_{H^{2\mu-\frac{2}{p}+1-\varepsilon,q}(\Omega)} + \|n\|_{H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega)}\|c\|_{H^{2\mu-\frac{2}{p}+1-\varepsilon,q}(\Omega)}\\
&\quad + \|u\|_{H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega)}\|n\|_{H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega)} + \|u\|_{H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega)}\|c\|_{H^{2\mu-\frac{2}{p}+1-\varepsilon,q}(\Omega)}\\
&\quad + \|u\|_{H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega)}\|u\|_{H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega)}\Big).
\end{aligned} \tag{3.28}$$

Since the trace space $X_{\gamma,\mu}$ is a Besov space, we will need embeddings concerning Besov spaces in the following. Since $1 < q \leq 2$, from [77, Theorem 4.6.1(a),(b)] we obtain

$$
\begin{aligned}
&B_{qp}^{2\mu-2/p}(\Omega) \hookrightarrow B_{qq}^{2\mu-2/p-\varepsilon}(\Omega) \hookrightarrow H^{2\mu-2/p-\varepsilon,q}(\Omega),\\
&B_{qp}^{2\mu-2/p+1}(\Omega) \hookrightarrow B_{qq}^{2\mu-2/p+1-\varepsilon}(\Omega) \hookrightarrow H^{2\mu-2/p+1-\varepsilon,q}(\Omega)
\end{aligned}
\tag{3.29}
$$

By applying these embeddings to (3.28) we obtain

$$
\begin{aligned}
\|F(w)\|_{X_0} &\leq c_3\Big(\|n\|_{B_{qp}^{2\mu-2/p}}\|c\|_{B_{qp}^{2\mu-2/p+1}(\Omega)} + \|u\|_{B_{qp}^{2\mu-2/p}}\|n\|_{B_{qp}^{2\mu-2/p}}\\
&\qquad + \|u\|_{B_{qp}^{2\mu-2/p}}\|c\|_{B_{qp}^{2\mu-2/p+1}(\Omega)} + \|u\|^2_{B_{qp}^{2\mu-2/p}}\Big)\\
&\leq c_4\|w\|^2_{X_{\gamma,\mu}}.
\end{aligned}
$$

This proves b) provided (3.24). In the case of (3.25), from [77, Theorem 4.6.1(e)] we obtain the embeddings

$$
\begin{aligned}
&B_{qp}^{2\mu-2/p}(\Omega) \hookrightarrow C^1(\overline{\Omega}),\\
&B_{qp}^{2\mu-2/p+1}(\Omega) \hookrightarrow C^2(\overline{\Omega})
\end{aligned}
\tag{3.30}
$$

Applying these in (3.26) we obtain assertion b) in this case.

It remains to verify assertion c). To this end, let $w, \tilde{w} \in \overline{B_{X_{\gamma,\mu}}}(0, R)$. We obtain

$$
\begin{aligned}
&\|F(w) - F(\tilde{w})\|_{X_0}\\
&\leq \|-\nabla\cdot(n\nabla c) - u\cdot\nabla n + \nabla\cdot(\tilde{n}\nabla\tilde{c}) + \tilde{u}\cdot\nabla\tilde{n}\|_{L^q(\Omega)}\\
&\quad + \|-u\cdot\nabla c + \tilde{u}\cdot\nabla\tilde{c}\|_{W^{1,q}(\Omega)} + \|\mathbb{P}[\kappa(u\cdot\nabla)u] - \mathbb{P}[\kappa(\tilde{u}\cdot\nabla)\tilde{u}]\|_{L^q_\sigma(\Omega)}\\
&\leq \|\nabla n\nabla c - \nabla\tilde{n}\nabla\tilde{c}\|_{L^q(\Omega)} + \|n\Delta c - \tilde{n}\Delta\tilde{c}\|_{L^q(\Omega)}\\
&\quad + \|u\cdot\nabla n - \tilde{u}\cdot\nabla\tilde{n}\|_{L^q(\Omega)} + \|u\cdot\nabla c - \tilde{u}\cdot\nabla\tilde{c}\|_{W^{1,q}(\Omega)}\\
&\quad + \|\kappa(u\cdot\nabla)u - \kappa(\tilde{u}\cdot\nabla)\tilde{u}\|_{L^q(\Omega)}\\
&\leq \|\nabla(n-\tilde{n})\nabla c\|_{L^q(\Omega)} + \|\nabla\tilde{n}\nabla(c-\tilde{c})\|_{L^q(\Omega)} + \|(n-\tilde{n})\Delta c\|_{L^q(\Omega)}\\
&\quad + \|\tilde{n}\Delta(c-\tilde{c})\|_{L^q(\Omega)} + \|(u-\tilde{u})\cdot\nabla n\|_{L^q(\Omega)} + \|\tilde{u}\cdot\nabla(n-\tilde{n})\|_{L^q(\Omega)}\\
&\quad + \|(u-\tilde{u})\cdot\nabla c\|_{W^{1,q}(\Omega)} + \|\tilde{u}\cdot\nabla(c-\tilde{c})\|_{W^{1,q}(\Omega)}\\
&\quad + \|\kappa((u-\tilde{u})\cdot\nabla)u\|_{L^q(\Omega)} + \|\kappa(\tilde{u}\cdot\nabla)(u-\tilde{u})\|_{L^q(\Omega)}
\end{aligned}
$$

Next, we use Hölder's inequality and Sobolev embeddings to obtain

$$
\begin{aligned}
&\|F(w) - F(\tilde{w})\|_{X_0} \\
&\quad \le \|\nabla(n-\tilde{n})\|_{L^{2q}(\Omega)}\|\nabla c\|_{L^{2q}(\Omega)} + \|\nabla\tilde{n}\|_{L^{2q}(\Omega)}\|\nabla(c-\tilde{c})\|_{L^{2q}(\Omega)} \\
&\qquad + \|n-\tilde{n}\|_{L^{2q}(\Omega)}\|\Delta c\|_{L^{2q}(\Omega)} + \|\tilde{n}\|_{L^{2q}(\Omega)}\|\Delta(c-\tilde{c})\|_{L^{2q}(\Omega)} \\
&\qquad + \|u-\tilde{u}\|_{L^{2q}(\Omega)}\|\nabla n\|_{L^{2q}(\Omega)} + \|\tilde{u}\|_{L^{2q}(\Omega)}\|\nabla(n-\tilde{n})\|_{L^{2q}(\Omega)} \\
&\qquad + c_0(\|u-\tilde{u}\|_{W^{1,2q}(\Omega)}\|\nabla c\|_{W^{1,2q}(\Omega)} + \|\tilde{u}\|_{W^{1,2q}(\Omega)}\|\nabla(c-\tilde{c})\|_{W^{1,2q}(\Omega)}) \\
&\qquad + |\kappa|\|u-\tilde{u}\|_{L^{2q}(\Omega)}\|\nabla u\|_{L^{2q}(\Omega)} + |\kappa|\|\tilde{u}\|_{L^{2q}(\Omega)}\|\nabla(u-\tilde{u})\|_{L^{2q}(\Omega)} \\
&\quad \le c_1\Big(\|n-\tilde{n}\|_{W^{1,2q}(\Omega)}\|c\|_{W^{1,2q}(\Omega)} + \|\tilde{n}\|_{W^{1,2q}(\Omega)}\|c-\tilde{c}\|_{W^{1,2q}(\Omega)} \\
&\qquad + \|n-\tilde{n}\|_{L^{2q}(\Omega)}\|c\|_{W^{2,2q}(\Omega)} + \|\tilde{n}\|_{L^{2q}(\Omega)}\|c-\tilde{c}\|_{W^{2,2q}(\Omega)} \\
&\qquad + \|u-\tilde{u}\|_{L^{2q}(\Omega)}\|n\|_{W^{1,2q}(\Omega)} + \|\tilde{u}\|_{L^{2q}(\Omega)}\|n-\tilde{n}\|_{W^{1,2q}(\Omega)} \\
&\qquad + \|u-\tilde{u}\|_{W^{1,2q}(\Omega)}\|c\|_{W^{2,2q}(\Omega)} + \|\tilde{u}\|_{W^{1,2q}(\Omega)}\|c-\tilde{c}\|_{W^{2,2q}(\Omega)} \\
&\qquad + \|u-\tilde{u}\|_{L^{2q}(\Omega)}\|u\|_{W^{1,2q}(\Omega)} + \|\tilde{u}\|_{L^{2q}(\Omega)}\|u-\tilde{u}\|_{W^{1,2q}(\Omega)}\Big).
\end{aligned}
$$

Similarly to the proof of b), for assumption (3.24) we use the Sobolev and Besov embeddings (3.27) and (3.29) to obtain

$$
\begin{aligned}
&\|F(w) - F(\tilde{w})\|_{X_0} \\
&\quad \le c_2\Big(\|n-\tilde{n}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\|c\|_{B^{2\mu-2/p+1}_{qp}(\Omega)} + \|\tilde{n}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\|c-\tilde{c}\|_{B^{2\mu-2/p+1}_{qp}(\Omega)} \\
&\qquad + \|u-\tilde{u}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\|n\|_{B^{2\mu-2/p}_{qp}(\Omega)} + \|\tilde{u}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\|n-\tilde{n}\|_{B^{2\mu-2/p}_{qp}(\Omega)} \\
&\qquad + \|u-\tilde{u}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\|c\|_{B^{2\mu-2/p+1}_{qp}(\Omega)} + \|\tilde{u}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\|c-\tilde{c}\|_{B^{2\mu-2/p+1}_{qp}(\Omega)} \\
&\qquad + \|u-\tilde{u}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\|u\|_{B^{2\mu-2/p}_{qp}(\Omega)} + \|\tilde{u}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\|u-\tilde{u}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\Big) \\
&\quad \le c_3(\|w\|_{X_{\gamma,\mu}} + \|\tilde{w}\|_{X_{\gamma,\mu}})\|w-\tilde{w}\|_{X_{\gamma,\mu}} \\
&\quad \le c_4 R\|w-\tilde{w}\|_{X_{\gamma,\mu}}.
\end{aligned}
$$

As in the proof of b), for assumption (3.25) we use instead the embeddings (3.30). This proves c). Hence, the proof is complete. □

3.2.2 Weak Setting

Let $p, q \in (1,\infty)$ and $\mu \in (1/p, 1]$. For the weak setting as underlying ground space we choose the space $X_0 = W^{-1,q}(\Omega) \times L^q(\Omega) \times L^q_\sigma(\Omega)$. Furthermore, we define the time-weighted solution and data spaces by

$$\begin{aligned}
&\mathbb{E}_{1,\mu} := L^p_\mu(0,T;W^{1,q}(\Omega)) \cap H^{1,p}_\mu(0,T;W^{-1,q}(\Omega)),\\
(3.31)\quad &\mathbb{E}_{2,\mu} := L^p_\mu(0,T;W^{2,q}_N(\Omega)) \cap H^{1,p}_\mu(0,T;L^q(\Omega)),\\
&\mathbb{E}_{3,\mu} := L^p_\mu(0,T;W^{2,q}(\Omega)\cap W^{1,q}_0(\Omega)\cap L^q_\sigma(\Omega)) \cap H^{1,p}_\mu(0,T;L^q_\sigma(\Omega)),
\end{aligned}$$

as well as

$$\mathbb{F}_\mu := L^p_\mu(0,T;X_0) \quad \text{and} \quad \mathbb{E}_\mu := \mathbb{E}_{1,\mu}\times\mathbb{E}_{2,\mu}\times\mathbb{E}_{3,\mu}. \tag{3.32}$$

By real interpolation, we obtain for the trace space, see, e.g., [5] and [72],

$$X_{\gamma,\mu} = B^{2\mu-2/p-1}_{qp}(\Omega)\times B^{2\mu-2/p}_{qp,N}(\Omega)\times B^{2\mu-2/p}_{qp,0}(\Omega)\cap L^q_\sigma(\Omega). \tag{3.33}$$

In order to apply the abstract theory of Subsection 2.5.2 to the coupled chemotaxis-Navier–Stokes system (IVKSNS), on X_0 we define the operator $\mathcal{A}$ and the mapping F as

$$\mathcal{A} := \begin{pmatrix} \Delta_{N,w} & 0 & 0\\ 1 & \Delta_N - 1 & 0\\ \mathbb{P}\nabla\phi & 0 & A_D \end{pmatrix}, \quad F(w) := \begin{pmatrix} -\nabla\cdot(n\nabla c) - u\cdot\nabla n\\ -u\cdot\nabla c\\ \mathbb{P}\kappa(u\cdot\nabla)u \end{pmatrix}$$

for $w=(n,c,u)^T$. Here $\Delta_{N,w}$ with domain $D(\Delta_{N,w}) = W^{1,q}(\Omega)$ denotes the Neumann–Laplacian on $W^{-1,q}(\Omega)$ and Δ_N with domain $D(\Delta_N) = W^{2,q}_N(\Omega)$ denotes the Neumann–Laplacian on $L^q(\Omega)$. Furthermore, as before A_D denotes the Stokes operator on $L^q_\sigma(\Omega)$. Then, using this notation and the Helmholtz projection $\mathbb{P}$, the system (IVKSNS) corresponds again to the abstract equation (3.23)

Assuming for p, q, and μ the same conditions as in the strong setting, namely, assuming that either

$$\begin{aligned}
&p,q\in(1,\infty),\ \mu\in(1/p,1] \quad \text{such that}\\
&\frac{d}{2} < q \le 2 \quad \text{and} \quad \frac{2}{p}+\frac{d}{2q} < 2\mu-1
\end{aligned} \tag{3.34}$$

or

$$p,q\in(1,\infty),\ \mu\in(1/p,1] \quad \text{such that} \quad \frac{2}{p}+\frac{d}{q} < 2\mu-1 \tag{3.35}$$

holds, our result on the existence of local strong solutions to (IVKSNS) in the weak setting reads as follows.

Theorem 3.2.2. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded domain with smooth boundary and assume that either* (3.34) *or* (3.35) *is satisfied. Moreover, let X_0, $\mathbb{E}_\mu$, $\mathbb{F}_\mu$, and $X_{\gamma,\mu}$ be defined as in* (3.31), (3.32), *and* (3.33). *Let $w_0 = (n_0, c_0, u_0)^T \in X_{\gamma,\mu}$.*

Then there exists $T = T(w_0) > 0$ such that (IVKSNS) *admits a unique solution $w = (n, c, u)^T \in \mathbb{E}_\mu$. In addition, for each $\delta \in (0, T)$ we have $w \in C([\delta, T]; X_\gamma)$, i.e., the solution regularizes instantaneously.*

Proof. In order to prove the theorem, by Proposition 2.5.3 we need to show the following properties:

a) The operator $\mathcal{A}$ admits maximal L^p-regularity on X_0.

b) $F(w) \in \mathrm{X}_0$ for any $w \in X_{\gamma,\mu}$.

c) There exists a $C > 0$ such that for any $R > 0$ it holds

$$\|F(w) - F(\tilde{w})\|_{X_0} \leq CR\|w - \tilde{w}\|_{X_{\gamma,\mu}}$$

for all $w, \tilde{w} \in \overline{B_{X_{\gamma,\mu}}}(0, R)$.

First we prove a). Due to the results from Section 2.4, the operators $\Delta_{N,w}$ on $W^{-1,q}(\Omega)$, $(\Delta_N - 1)$ on $L^q(\Omega)$, and A_D on $L^q_\sigma(\Omega)$ admit maximal L^p-regularity. Hence, as in the strong setting due to the triangular structure of $\mathcal{A}$, we see that $\mathcal{A}$ admits maximal L^p-regularity on X_0.

Next, we show that $F(w(t)) \in X_0$ is satisfied for $w \in X_{\gamma,\mu}$. Using the fact that u is divergence-free and that ∇ is a bounded operator from $L^q(\Omega) = (L^{q'}(\Omega))'$ to $(W^{1,q'}(\Omega))'$, for $w \in X_{\gamma,\mu}$ we obtain

$$\begin{aligned}
&\|F(w)\|_{X_0} \\
&\leq \|\nabla \cdot (n\nabla c) + \nabla \cdot (un)\|_{W^{-1,q}(\Omega)} + \|u \cdot \nabla c\|_{L^q(\Omega)} + \|\mathbb{P}[\kappa(u \cdot \nabla)u]\|_{L^q_\sigma(\Omega)} \\
&\leq c_1(\|n\nabla c\|_{L^q(\Omega)} + \|un\|_{L^q(\Omega)}) + \|u \cdot \nabla c\|_{L^q(\Omega)} + \|\mathbb{P}[\kappa(u \cdot \nabla)u]\|_{L^q_\sigma(\Omega)}.
\end{aligned}$$

Then, Hölder's inequality yields

$$\begin{aligned}
\|F(w)\|_{X_0} &\leq c_2\Big(\|n\|_{L^{2q}(\Omega)}\|\nabla c\|_{L^{2q}(\Omega)} + \|u\|_{L^{2q}(\Omega)}\|n\|_{L^{2q}(\Omega)} \\
&\quad + \|u\|_{L^{2q}(\Omega)}\|\nabla c\|_{L^{2q}(\Omega)} + \|u\|_{L^{2q}(\Omega)}\|\nabla u\|_{L^{2q}(\Omega)}\Big) \\
&\leq c_3\Big(\|n\|_{L^{2q}(\Omega)}\|c\|_{W^{1,2q}(\Omega)} + \|u\|_{L^{2q}(\Omega)}\|n\|_{L^{2q}(\Omega)} \\
&\quad + \|u\|_{L^{2q}(\Omega)}\|c\|_{W^{1,2q}(\Omega)} + \|u\|_{L^{2q}(\Omega)}\|u\|_{W^{1,2q}(\Omega)}\Big).
\end{aligned} \tag{3.36}$$

Assuming (3.34), we have the following Sobolev embeddings

$$
\begin{aligned}
H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega) &\hookrightarrow W^{1,2q}(\Omega),\\
H^{2\mu-\frac{2}{p}-1-\varepsilon,q}(\Omega) &\hookrightarrow L^{2q}(\Omega).
\end{aligned}
\tag{3.37}
$$

Using these embeddings, we obtain

$$
\begin{aligned}
&\|F(w)\|_{X_0}\\
&\quad\le c_4\Big(\|n\|_{H^{2\mu-\frac{2}{p}-1-\varepsilon,q}(\Omega)}\|c\|_{H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega)}+\|u\|_{H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega)}\|n\|_{H^{2\mu-\frac{2}{p}-1-\varepsilon,q}(\Omega)}\\
&\qquad+\|u\|_{H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega)}\|c\|_{H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega)}+\|u\|_{H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega)}\|u\|_{H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega)}\Big).
\end{aligned}
$$

Furthermore, since $1<q\le 2$, due to [77, Theorem 4.6.1(a),(b)], we have

$$
\begin{aligned}
B_{qp}^{2\mu-2/p}(\Omega) &\hookrightarrow B_{qq}^{2\mu-2/p-\varepsilon}(\Omega)\hookrightarrow H^{2\mu-2/p-\varepsilon,q}(\Omega),\\
B_{qp}^{2\mu-2/p-1}(\Omega) &\hookrightarrow B_{qq}^{2\mu-2/p-1-\varepsilon}(\Omega)\hookrightarrow H^{2\mu-2/p-1-\varepsilon,q}(\Omega)
\end{aligned}
\tag{3.38}
$$

Then, by applying these embeddings, we obtain

$$
\begin{aligned}
&\|F(w)\|_{X_0}\\
&\quad\le c_5\Big(\|n\|_{B_{qp}^{2\mu-2/p-1}(\Omega)}\|c\|_{B_{qp}^{2\mu-2/p}(\Omega)}+\|u\|_{B_{qp}^{2\mu-2/p}(\Omega)}\|n\|_{B_{qp}^{2\mu-2/p-1}(\Omega)}\\
&\qquad+\|u\|_{B_{qp}^{2\mu-2/p}(\Omega)}\|c\|_{B_{qp}^{2\mu-2/p}(\Omega)}+\|u\|_{B_{qp}^{2\mu-2/p}(\Omega)}\|u\|_{B_{qp}^{2\mu-2/p}(\Omega)}\Big)\\
&\quad\le 4c_5\|w\|_{X_{\gamma,\mu}}^2.
\end{aligned}
$$

This proves b) provided (3.34). In the case of (3.25), from [77, Theorem 4.6.1(e)] we obtain the embeddings

$$
\begin{aligned}
B_{qp}^{2\mu-2/p}(\Omega) &\hookrightarrow C^1(\overline{\Omega}),\\
B_{qp}^{2\mu-2/p-1}(\Omega) &\hookrightarrow C^0(\overline{\Omega}).
\end{aligned}
\tag{3.39}
$$

Applying these in (3.36), we obtain assertion b) in this case.

It remains to verify assertion c). To this end, let $w,\tilde{w}\in\overline{B_{X_{\gamma,\mu}}}(0,R)$. As above, using that u is divergence-free and that ∇ is a bounded operator from $L^q(\Omega) = (L^{q'}(\Omega))'$ to $(W^{1,q'}(\Omega))'$, we obtain

$$\begin{aligned}
&\|F(w) - F(\tilde{w})\|_{X_0} \\
&\quad \le \|\nabla \cdot (n\nabla c) - \nabla \cdot (\tilde{n}\nabla\tilde{c})\|_{W^{-1,q}(\Omega)} + \|u \cdot \nabla n - \tilde{u} \cdot \nabla\tilde{n}\|_{W^{-1,q}(\Omega)} \\
&\qquad + \|u \cdot \nabla c - \tilde{u} \cdot \nabla\tilde{c}\|_{L^q(\Omega)} + \|\mathbb{P}\kappa((u \cdot \nabla)u) - \mathbb{P}\kappa((\tilde{u} \cdot \nabla)\tilde{u})\|_{L^q_\sigma(\Omega)} \\
&\quad \le \|\nabla \cdot (n\nabla c - \tilde{n}\nabla\tilde{c})\|_{W^{-1,q}(\Omega)} + \|\nabla \cdot (un - \tilde{u}\tilde{n})\|_{W^{-1,q}(\Omega)} \\
&\qquad + \|u \cdot \nabla c - \tilde{u} \cdot \nabla\tilde{c}\|_{L^q(\Omega)} + \|\kappa(u \cdot \nabla)u - \kappa(\tilde{u} \cdot \nabla)\tilde{u}\|_{L^q(\Omega)} \\
&\quad \le c_1(\|n\nabla c - \tilde{n}\nabla\tilde{c}\|_{L^q(\Omega)} + \|un - \tilde{u}\tilde{n}\|_{L^q(\Omega)}) \\
&\qquad + \|u \cdot \nabla c - \tilde{u} \cdot \nabla\tilde{c}\|_{L^q(\Omega)} + \|\kappa(u \cdot \nabla)u - \kappa(\tilde{u} \cdot \nabla)\tilde{u}\|_{L^q(\Omega)} \\
&\quad \le c_1\Big(\|(n - \tilde{n})\nabla c\|_{L^q(\Omega)} + \|\tilde{n}\nabla(c - \tilde{c})\|_{L^q(\Omega)} \\
&\qquad + \|(u - \tilde{u})n\|_{L^q(\Omega)} + \|\tilde{u}(n - \tilde{n})\|_{L^q(\Omega)}\Big) \\
&\qquad + \|(u - \tilde{u}) \cdot \nabla c\|_{L^q(\Omega)} + \|\tilde{u} \cdot \nabla(c - \tilde{c})\|_{L^q(\Omega)} \\
&\qquad + \|\kappa((u - \tilde{u}) \cdot \nabla)u\|_{L^q(\Omega)} + \|\kappa(\tilde{u} \cdot \nabla)(u - \tilde{u})\|_{L^q(\Omega)}.
\end{aligned}$$

Next, we use Hölder's inequality to obtain

$$\begin{aligned}
&\|F(w) - F(\tilde{w})\|_{X_0} \\
&\quad \le c_2\Big(\|n - \tilde{n}\|_{L^{2q}(\Omega)}\|c\|_{W^{1,2q}(\Omega)} + \|\tilde{n}\|_{L^{2q}(\Omega)}\|c - \tilde{c}\|_{W^{1,2q}(\Omega)} \\
&\qquad + \|u - \tilde{u}\|_{L^{2q}(\Omega)}\|n\|_{L^{2q}(\Omega)} + \|\tilde{u}\|_{L^{2q}(\Omega)}\|n - \tilde{n}\|_{L^{2q}(\Omega)} \\
&\qquad + \|u - \tilde{u}\|_{L^{2q}(\Omega)}\|c\|_{W^{1,2q}(\Omega)} + \|\tilde{u}\|_{L^{2q}(\Omega)}\|c - \tilde{c}\|_{W^{1,2q}(\Omega)} \\
&\qquad + \|u - \tilde{u}\|_{L^{2q}(\Omega)}\|u\|_{W^{1,2q}(\Omega)} + \|\tilde{u}\|_{L^{2q}(\Omega)}\|u - \tilde{u}\|_{W^{1,2q}(\Omega)}\Big).
\end{aligned}$$

Similarly to the proof of b), for assumption (3.34) we use the Sobolev and Besov embeddings (3.37) and (3.38) to obtain

$$\begin{aligned}
&\|F(w) - F(\tilde{w})\|_{X_0} \\
&\quad \le c_3\Big(\|n - \tilde{n}\|_{B^{2\mu-2/p-1}_{qp}(\Omega)}\|c\|_{B^{2\mu-2/p}_{qp}(\Omega)} + \|\tilde{n}\|_{B^{2\mu-2/p-1}_{qp}(\Omega)}\|c - \tilde{c}\|_{B^{2\mu-2/p}_{qp}(\Omega)} \\
&\qquad + \|u - \tilde{u}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\|n\|_{B^{2\mu-2/p-1}_{qp}(\Omega)} + \|\tilde{u}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\|n - \tilde{n}\|_{B^{2\mu-2/p-1}_{qp}(\Omega)} \\
&\qquad + \|u - \tilde{u}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\|c\|_{B^{2\mu-2/p}_{qp}(\Omega)} + \|\tilde{u}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\|c - \tilde{c}\|_{B^{2\mu-2/p}_{qp}(\Omega)} \\
&\qquad + \|u - \tilde{u}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\|u\|_{B^{2\mu-2/p}_{qp}(\Omega)} + \|\tilde{u}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\|u - \tilde{u}\|_{B^{2\mu-2/p}_{qp}(\Omega)}\Big) \\
&\quad \le c_4(\|w\|_{X_{\gamma,\mu}} + \|\tilde{w}\|_{X_{\gamma,\mu}})\|w - \tilde{w}\|_{X_{\gamma,\mu}} \\
&\quad \le c_5 R\|w - \tilde{w}\|_{X_{\gamma,\mu}}.
\end{aligned}$$

As in the proof of b), for assumption (3.35) we use instead the embeddings (3.39). This proves c). Hence, the proof is complete. □

CHAPTER 4

The Keller–Segel Model on Convex Domains

In the last chapter we considered a coupled chemotaxis-fluid system on smooth domains. Now, we go a step backwards and consider the Keller–Segel model without coupling to a fluid, but this time on a more general domain. Namely, in this chapter the physical domain $\Omega \subset \mathbb{R}^3$ is supposed to be a bounded and convex domain. Therefore, recall the *classical* Keller–Segel system of the form

$$\text{(KS)} \qquad \begin{cases} \partial_t n = \Delta n - \nabla \cdot (n \nabla c) & \text{in } (0,\infty) \times \Omega, \\ \partial_t c = \Delta c - c + n & \text{in } (0,\infty) \times \Omega, \\ \dfrac{\partial n}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = 0 & \text{on } (0,\infty) \times \partial\Omega, \end{cases}$$

where n and c denote the cell (or organism) density and the concentration of the chemical signal (e.g. oxygen concentration), respectively. The physical domain $\Omega \subset \mathbb{R}^3$ is supposed to be a bounded and convex domain. Furthermore, ν denotes the outward unit normal vector on $\partial\Omega$.

The aim of this chapter is to extend the results by Hieber and Stinner [44] to bounded convex domains. Furthermore, we show local well-posedness in this non-smooth setting and prove the existence of global solutions near equilibria.

This chapter is structured similar as the last one. First, in Section 4.1 we prove the existence and uniqueness of strong time-periodic solutions to the periodic Keller–Segel model. Then, in Section 4.2 we show local well-posedness. Furthermore, we prove the existence of global solutions provided the initial value is chosen close to an equilibrium.

4.1 The Time-Periodic Problem

In this section, we want to consider time-periodic solutions to the time-periodic Keller–Segel model

$$\text{(PKS)} \qquad \begin{cases} \partial_t n = \Delta n - \nabla \cdot (n \nabla c) + f_1(t) & \text{in } \mathbb{R} \times \Omega, \\ \partial_t c = \Delta c - c + n + f_2(t) & \text{in } \mathbb{R} \times \Omega, \\ \dfrac{\partial n}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ n(x,0) = n(x,T) & \text{in } \Omega, \\ c(x,0) = c(x,T) & \text{in } \Omega, \end{cases}$$

provided the T-periodic external forces f_1 and f_2 are sufficiently small. To this end, for $p \in (1,\infty)$, $q \in (1,2]$, and $q' \in (1,\infty)$ with $\frac{1}{q} + \frac{1}{q'} = 1$ we define the ground space X_0 and the solution spaces for n and c by

$$\text{(4.1)} \qquad \begin{aligned} X_0 &:= \left(W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega)\right)' \times L^q(\Omega), \\ \mathbb{E}_1 &:= L^p(0,T; W^{1,q}(\Omega) \cap L^q_{av}(\Omega)) \cap W^{1,p}(0,T; (W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'), \\ \mathbb{E}_2 &:= L^p(0,T; W^{2,q}_N(\Omega)) \cap W^{1,p}(0,T; L^q(\Omega)), \end{aligned}$$

respectively. Furthermore, we set

$$\text{(4.2)} \qquad \mathbb{F} := L^p(0,T; X_0) \text{ and } \mathbb{E} := \mathbb{E}_1 \times \mathbb{E}_2.$$

We want to apply the abstract theory introduced in Section 2.5.3. In order to do so, we define on X_0 the operator $\mathcal{A}$ and, for $w = (n,c)^T$, the mapping F as

$$\mathcal{A} := \begin{pmatrix} \Delta_{N,w} & 0 \\ 1 & \Delta_N - 1 \end{pmatrix}, \tag{4.3}$$

$$F(t,w) := \begin{pmatrix} -\nabla \cdot (n\nabla c) + f_1(t) \\ f_2(t) \end{pmatrix}, \tag{4.4}$$

where $\Delta_{N,w}$ with domain $D(\Delta_{N,w}) = W^{1,q}(\Omega) \cap L^q_{av}(\Omega)$ denotes the Neumann Laplacian on $(W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'$ and Δ_N with domain $D(\Delta_N) = W^{2,q}_N(\Omega)$ denotes the Neumann Laplacian on $L^q(\Omega)$.

Then, the Keller–Segel system (PKS) in the periodic setting corresponds to the equation

$$\begin{cases} \partial_t w(t) - \mathcal{A}w(t) = F(t, w(t)) & t \in (0,T), \\ w(0) = w(T). \end{cases} \tag{4.5}$$

Provided that

$$p, q \in (1,\infty) \quad \text{such that} \quad \frac{3}{2} < q \leq 2 \quad \text{and} \quad \frac{1}{p} + \frac{3}{2q} < 1 \tag{4.6}$$

our result on existence and uniqueness of strong time-periodic solutions to (PKS) reads as follows.

Theorem 4.1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain, $T > 0$, and assume that (4.6) is satisfied. Moreover, let X_0, $\mathbb{E}$, and $\mathbb{F}$ be defined as in (4.1) and (4.2) and let $f = (f_1, f_2)^T \in \mathbb{F}$ be T-periodic.*

Then there is $r_0 > 0$ such that for any $r \in (0, r_0)$ there exists $\delta = \delta(r) > 0$ such that if $\|f\|_{\mathbb{F}} < \delta$, then there exists a T-periodic solution $w = (n,c)^T \in \mathbb{E}$ to (PKS), which is unique in $\overline{B_{\mathbb{E}}}(0,r)$.

Proof. Due to Proposition 2.5.5 it suffices to show the following properties:

a) The operator $\mathcal{A}$ admits maximal periodic L^p-regularity on X_0.

b) $F(\cdot, w(\cdot)) \in \mathbb{F}$ for any $w \in \mathbb{E}$.

c) There exists a $C > 0$ such that for any $R > 0$ it holds

$$\|F(\cdot, w(\cdot)) - F(\cdot, \tilde{w}(\cdot))\|_{\mathbb{F}} \leq CR\|w - \tilde{w}\|_{\mathbb{E}}$$

for all $w, \tilde{w} \in \overline{B_{\mathbb{E}}}(0, R)$.

If a), b), and c) are satisfied, we may define C_R for $R \in (0, \frac{\delta_1}{C})$ by $C_R := CR$. Then, the assertion of the theorem follows from Proposition 2.5.5 for $\delta := \delta_2$ and $r_0 := \frac{\delta_1}{C}$.

Thus, it remains to prove the assertions a), b), and c).

First, we show assertion a). As introduced above, let $\Delta_{N,w}$ and Δ_N be the Neumann Laplacian on $(W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'$ and on $L^q(\Omega)$, respectively. For these two operators we have $0 \in \rho(\Delta_{N,w})$ and $0 \in \rho(\Delta_N - 1)$. Hence, the triangular structure of $\mathcal{A}$ implies that $0 \in \rho(\mathcal{A})$ holds true. Due to Proposition 2.4.6, $\Delta_{N,w}$ on $(W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'$ admits maximal L^p-regularity. Moreover, since Δ_N has the property of maximal L^p-regularity on $L^q(\Omega)$ (Proposition 2.4.4) as well, it follows again by the triangular structure that $\mathcal{A}$ admits maximal L^p-regularity on X_0. Using now Proposition 2.2.1 yields that $\mathcal{A}$ admits maximal periodic L^p-regularity on X_0, which proves a).

Next, we show that $F(\cdot, w(\cdot)) \in \mathbb{F}$ is satisfied for $w \in \mathbb{E}$. Using that ∇ is a bounded operator from $L^q(\Omega) = (L^q(\Omega))'$ to $(W^{1,q'}(\Omega))'$ as well as Hölder's inequality , we obtain for $w \in \mathbb{E}$ the estimate

$$\begin{aligned}
&\|F(\cdot, w(\cdot))\|_{\mathbb{F}} \\
&\quad \leq \| - \nabla \cdot (n\nabla c) + f_1\|_{L^p(0,T;(W^{1,q'}(\Omega)\cap L^{q'}_{av}(\Omega))')} + \|f_2\|_{L^p(0,T;L^q(\Omega))} \\
&\quad \leq c_1\|n\nabla c\|_{L^p(0,T;L^q(\Omega))} + \|f\|_{\mathbb{F}} \\
&\quad \leq c_1\|n\|_{L^{2p}(0,T;L^{2q}(\Omega)}\|\nabla c\|_{L^{2p}(0,T;L^{2q}(\Omega)} + \|f\|_{\mathbb{F}} \\
&\quad \leq c_1\|n\|_{L^{2p}(0,T;L^{2q}(\Omega)}\|c\|_{L^{2p}(0,T;W^{1,2q}(\Omega)} + \|f\|_{\mathbb{F}}.
\end{aligned} \tag{4.7}$$

Due to Proposition 2.5.2 we have

$$\mathbb{E}_1 \hookrightarrow H^{\theta,p}(0,T; H^{-1+2(1-\theta),q}(\Omega)),\ \mathbb{E}_2 \hookrightarrow H^{\theta,p}(0,T; H^{2(1-\theta),q}(\Omega)) \tag{4.8}$$

for any $\theta \in (0, 1)$. In view of (4.6), we deduce the existence of $\theta \in (0, 1)$ satisfying

$$\frac{1}{2p} < \theta < \frac{1}{2} - \frac{3}{4q}.$$

This inequality is equivalent to

$$\theta - \frac{1}{p} > -\frac{1}{2p}, \quad -1 + 2(1-\theta) - \frac{3}{q} > -\frac{3}{2q}.$$

Hence, by Sobolev embeddings (see, e.g., [2, Theorem 4.12] for non-smooth domains) we obtain

$$\begin{aligned} &H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega)) \hookrightarrow L^{2p}(0,T;L^{2q}(\Omega)) \\ &H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega)) \hookrightarrow L^{2p}(0,T;W^{1,2q}(\Omega)). \end{aligned} \tag{4.9}$$

Using the Sobolev embeddings (4.9) and the embeddings due to the mixed derivative theorem (4.8) in (4.7), we obtain

$$\begin{aligned} &\|F(\cdot,w(\cdot))\|_{\mathbb{F}} \\ &\quad \le c_2 \|n\|_{H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega))} \|c\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} + \|f\|_{\mathbb{F}} \\ &\quad \le c_3 \|n\|_{\mathbb{E}_1} \|c\|_{\mathbb{E}_2} + \|f\|_{\mathbb{F}} \\ &\quad \le c_3 \|w\|_{\mathbb{E}}^2 + \|f\|_{\mathbb{F}}. \end{aligned}$$

Furthermore, due to $(W^{1,q'}(\Omega))' \subset (W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))'$, we obtain that the first component of F belongs to $L^p(0,T;(W^{1,q'}(\Omega) \cap L^{q'}_{av}(\Omega))')$. Hence, we conclude that $F(\cdot,w(\cdot)) \in \mathbb{F}$. This proves b).

It remains to verify assertion c). In order to do so, let $w, \tilde{w} \in \overline{B_{\mathbb{E}}}(0,R)$. By the boundedness of ∇ from $L^q(\Omega) = (L^{q'}(\Omega))'$ to $(W^{1,q'}(\Omega))'$, we obtain

$$\begin{aligned} &\|F(\cdot,w(\cdot)) - F(\cdot,\tilde{w}(\cdot))\|_{\mathbb{F}} \\ &\quad \le \|\nabla \cdot (n\nabla c - \tilde{n}\nabla\tilde{c})\|_{L^p(0,T;(W^{1,q'}(\Omega)\cap L^{q'}_{av}(\Omega))')} \\ &\quad \le c_1 \|n\nabla c - \tilde{n}\nabla\tilde{c}\|_{L^p(0,T;L^q(\Omega))} \\ &\quad \le c_1 \left(\|(n-\tilde{n})\nabla c\|_{L^p(0,T;L^q(\Omega))} + \|\tilde{n}\nabla(c-\tilde{c})\|_{L^p(0,T;L^q(\Omega))} \right). \end{aligned}$$

Next, we use Hölder's inequality to obtain

$$\begin{aligned} &\|F(\cdot,w(\cdot)) - F(\cdot,\tilde{w}(\cdot))\|_{\mathbb{F}} \\ &\quad \le c_1 \Big(\|n-\tilde{n}\|_{L^{2p}(0,T;L^{2q}(\Omega))} \|\nabla c\|_{L^{2p}(0,T;L^{2q}(\Omega))} \\ &\qquad + \|\tilde{n}\|_{L^{2p}(0,T;L^{2q}(\Omega))} \|\nabla(c-\tilde{c})\|_{L^{2p}(0,T;L^{2q}(\Omega))} \Big) \\ &\quad \le c_1 \Big(\|n-\tilde{n}\|_{L^{2p}(0,T;L^{2q}(\Omega))} \|c\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} \\ &\qquad + \|\tilde{n}\|_{L^{2p}(0,T;L^{2q}(\Omega))} \|c-\tilde{c}\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} \Big). \end{aligned}$$

Finally, by the Sobolev embeddings (4.9) and the embeddings due to the mixed derivative theorem (4.8) we obtain

$$
\begin{aligned}
&\|F(\cdot, w(\cdot)) - F(\cdot, \tilde{w}(\cdot))\|_{\mathbb{F}} \\
&\quad \le c_2 \Big(\|n - \tilde{n}\|_{H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega))} \|c\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \\
&\qquad + \|\tilde{n}\|_{H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega))} \|c - \tilde{c}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \Big) \\
&\quad \le c_3 \Big(\|n - \tilde{n}\|_{\mathbb{E}_1} \|c\|_{\mathbb{E}_2} + \|\tilde{n}\|_{\mathbb{E}_1} \|c - \tilde{c}\|_{\mathbb{E}_2} \Big) \\
&\quad \le c_3 \left(\|w\|_{\mathbb{E}} + \|\tilde{w}\|_{\mathbb{E}} \right) \|w - \tilde{w}\|_{\mathbb{E}} \\
&\quad \le c_4 R \|w - \tilde{w}\|_{\mathbb{E}}.
\end{aligned}
$$

This proves c). Hence, the proof is complete. □

4.2 The Initial Value Problem

We consider the Keller–Segel initial value problem

$$
\text{(IVKS)} \qquad \begin{cases}
\partial_t n = \Delta n - \nabla \cdot (n \nabla c) & \text{in } (0,\infty) \times \Omega, \\
\partial_t c = \Delta c - c + n & \text{in } (0,\infty) \times \Omega, \\
\dfrac{\partial n}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = 0 & \text{on } (0,\infty) \times \partial\Omega, \\
n(x,0) = n_0(x) & \text{in } \Omega, \\
c(x,0) = c_0(x) & \text{in } \Omega
\end{cases}
$$

and want to show local well-posedness as well as the existence of global solutions provided the initial value is chosen close to an equilbrium by using the abstract theory introduced in Subsection 2.5.2. To this end, for $p \in (1,\infty)$, $q \in (1,2]$, and $\mu \in (1/p, 1]$ we define the time-weighted solution and data spaces

$$
\begin{aligned}
X_0 &:= W^{-1,q}(\Omega) \times L^q(\Omega), \\
\mathbb{E}_{1,\mu} &:= L^p_\mu(0,T;W^{1,q}(\Omega)) \cap H^{1,p}_\mu(0,T;W^{-1,q}(\Omega)), \qquad (4.10)\\
\mathbb{E}_{2,\mu} &:= L^p_\mu(0,T;W^{2,q}_N(\Omega)) \cap H^{1,p}_\mu(0,T;L^q(\Omega)),
\end{aligned}
$$

as well as

$$
\mathbb{F}_\mu := L^p_\mu(0,T;X_0) \text{ and } \mathbb{E}_\mu := \mathbb{E}_{1,\mu} \times \mathbb{E}_{2,\mu}. \qquad (4.11)
$$

By real interpolation we obtain for the trace space

$$X_{\gamma,\mu} = B_{qp}^{2\mu-2/p-1}(\Omega) \times B_{qp,N}^{2\mu-2/p}(\Omega). \tag{4.12}$$

Next, we define on X_0 the operator $\mathcal{A}$ and, for $w = (n,c)^T$, the nonlinearity F as

$$\mathcal{A} := \begin{pmatrix} \Delta_{N,w} & 0 \\ 1 & \Delta_N - 1 \end{pmatrix}, \tag{4.13}$$

$$F(t,w) := \begin{pmatrix} -\nabla \cdot (n\nabla c) \\ 0 \end{pmatrix}, \tag{4.14}$$

where $\Delta_{N,w}$ with domain $D(\Delta_{N,w}) = W^{1,q}(\Omega)$ denotes the Neumann Laplacian on $W^{-1,q}(\Omega)$ and Δ_N with domain $D(\Delta_N) = W_N^{2,q}(\Omega)$ denotes the Neumann Laplacian on $L^q(\Omega)$.

Then, the Keller–Segel system (IVKS) corresponds to the equation

$$\begin{cases} \partial_t w(t) - \mathcal{A}w(t) = F(w(t)) & t \in (0,\infty), \\ \qquad\qquad w(0) = w_0 \end{cases} \tag{4.15}$$

with $w_0 = (n_0, c_0)^T$.

Provided that

$$\begin{aligned} &p, q \in (1,\infty),\ \mu \in (1/p, 1] \quad \text{such that} \\ &\frac{3}{2} < q \le 2 \quad \text{and} \quad \frac{2}{p} + \frac{3}{2q} < 2\mu - 1 \end{aligned} \tag{4.16}$$

our result on existence of strong solutions to (IVKS) reads as follows.

Theorem 4.2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain and assume that* (4.16) *is satisfied. Moreover, let X_0, $\mathbb{E}_\mu$, $\mathbb{F}_\mu$, and $X_{\gamma,\mu}$ be defined as in* (4.10), (4.11), *and* (4.12). *Let $w_0 = (n_0, c_0)^T \in X_{\gamma,\mu}$.*

Then there exists $T = T(w_0) > 0$ such that (IVKS) *admits a unique solution $w = (n,c)^T \in \mathbb{E}_\mu$. In addition, for each $\delta \in (0,T)$ we have $w \in C([\delta,T]; X_\gamma)$, i.e., the solution regularizes instantaneously.*

Proof. In order to prove the theorem, by Proposition 2.5.3 it suffices to show the following properties:

a) The operator $\mathcal{A}$ admits maximal L^p-regularity on X_0.

b) $F(w) \in \mathrm{X}_0$ for any $w \in \mathrm{X}_{\gamma,\mu}$.

c) There exists a $C > 0$ such that for any $R > 0$ it holds

$$\|F(w) - F(\tilde{w})\|_{X_0} \leq CR\|w - \tilde{w}\|_{X_{\gamma,\mu}}$$

for all $w, \tilde{w} \in \overline{B_{X_{\gamma,\mu}}}(0, R)$.

First we verify assertion a). Due to Proposition 2.4.6, $\Delta_{N,w}$ on $W^{-1,q}(\Omega)$ admits maximal L^p-regularity. Moreover, since Δ_N has the property of maximal L^p-regularity on $L^q(\Omega)$ (Proposition 2.4.4), it follows by the triangular structure that $\mathcal{A}$ admits maximal L^p-regularity on X_0, which proves a).

Next, we show that $F(w) \in X_0$ is satisfied for $w \in \mathrm{X}_{\gamma,\mu}$. Using the fact that ∇ is a bounded operator from $L^q(\Omega) = (L^{q'}(\Omega))'$ to $(W^{1,q'}(\Omega))'$ as well as Hölder's inequality, for $w \in \mathrm{X}_{\gamma,\mu}$ we obtain

$$\begin{aligned} \|F(w)\|_{X_0} &\leq \| - \nabla \cdot (n\nabla c)\|_{(W^{1,q'}(\Omega))'} \\ &\leq c_1\|n\nabla c\|_{L^q(\Omega)} \\ &\leq c_1\|n\|_{L^{2q}(\Omega)}\|\nabla c\|_{L^{2q}(\Omega)} \\ &\leq c_1\|n\|_{L^{2q}(\Omega)}\|c\|_{W^{1,2q}(\Omega)}. \end{aligned} \tag{4.17}$$

Due to (4.16) we have the following Sobolev embeddings

$$\begin{aligned} H^{2\mu-\frac{2}{p}-1-\varepsilon,q}(\Omega) &\hookrightarrow L^{2q}(\Omega), \\ H^{2\mu-\frac{2}{p}-\varepsilon,q}(\Omega) &\hookrightarrow W^{1,2q}(\Omega). \end{aligned} \tag{4.18}$$

Hence, using these embeddings we obtain

$$\|F(w)\|_{X_0} \leq c_2\|n\|_{H^{2\mu-2/p-1-\varepsilon,q}(\Omega)}\|c\|_{H^{2\mu-2/p-\varepsilon,q}(\Omega)}. \tag{4.19}$$

Since the trace space $X_{\gamma,\mu}$ is a Besov space, we will need embeddings concerning Besov spaces in the following. For details we refer to the monograph of Triebel [77]. Due to [77, Theorem 4.6.1(a),(b)] we obtain

$$\begin{aligned} &B_{qp}^{2\mu-2/p-1}(\Omega) \hookrightarrow B_{qq}^{2\mu-2/p-1-\varepsilon}(\Omega) \hookrightarrow H^{2\mu-2/p-1-\varepsilon,q}(\Omega) \\ &B_{qp}^{2\mu-2/p}(\Omega) \hookrightarrow B_{qq}^{2\mu-2/p-\varepsilon}(\Omega) \hookrightarrow H^{2\mu-2/p-\varepsilon,q}(\Omega) \end{aligned} \tag{4.20}$$

Then, by applying these embeddings to (4.19) we obtain

$$\begin{aligned} \|F(w)\|_{X_0} &\leq c_3 \|n\|_{B_{qp}^{2\mu-2/p-1}(\Omega)} \|c\|_{B_{qp}^{2\mu-2/p}(\Omega)} \\ &\leq c_3 \|w\|^2_{X_{\gamma,\mu}}. \end{aligned}$$

This proves b).

It remains to verify assertion c). To this end, let $w, \tilde{w} \in \overline{B_{X_{\gamma,\mu}}}(0,R)$. By the boundedness of ∇ from $L^q(\Omega) = (L^{q'}(\Omega))'$ to $(W^{1,q'}(\Omega))'$, we obtain

$$\begin{aligned} \|F(w) - F(\tilde{w})\|_{X_0} &\leq \|\nabla \cdot (n \nabla c - \tilde{n} \nabla \tilde{c})\|_{(W^{1,q'}(\Omega) \cap L_0^{q'}(\Omega))'} \\ &\leq c_1 \|n \nabla c - \tilde{n} \nabla \tilde{c}\|_{L^q(\Omega)} \\ &\leq c_1 \left(\|(n - \tilde{n}) \nabla c\|_{L^q(\Omega)} + \|\tilde{n} \nabla (c - \tilde{c})\|_{L^q(\Omega)} \right). \end{aligned}$$

Next, we use Hölder's inequality to obtain

$$\begin{aligned} &\|F(w) - F(\tilde{w})\|_{X_0} \\ &\quad \leq c_1 \Big(\|n - \tilde{n}\|_{L^{2q}(\Omega)} \|\nabla c\|_{L^{2q}(\Omega)} + \|\tilde{n}\|_{L^{2q}(\Omega)} \|\nabla (c - \tilde{c})\|_{L^{2q}(\Omega)} \Big) \\ &\quad \leq c_1 \Big(\|n - \tilde{n}\|_{L^{2q}(\Omega)} \|c\|_{W^{1,2q}(\Omega)} + \|\tilde{n}\|_{L^{2q}(\Omega)} \|c - \tilde{c}\|_{W^{1,2q}(\Omega)} \Big). \end{aligned}$$

Then, by using the embeddings (4.18) and (4.20) we have

$$\begin{aligned} &\|F(w) - F(\tilde{w})\|_{X_0} \\ &\quad \leq c_2 \Big(\|n - \tilde{n}\|_{H^{2\mu-2/p-1-\varepsilon,q}(\Omega)} \|c\|_{H^{2\mu-2/p-\varepsilon,q}(\Omega)} \\ &\qquad + \|\tilde{n}\|_{H^{2\mu-2/p-1-\varepsilon,q}(\Omega)} \|c - \tilde{c}\|_{H^{2\mu-2/p-\varepsilon,q}(\Omega)} \Big) \\ &\quad \leq c_3 \Big(\|n - \tilde{n}\|_{B_{qp}^{2\mu-2/p-1}(\Omega)} \|c\|_{B_{qp}^{2\mu-2/p}(\Omega)} \\ &\qquad + \|\tilde{n}\|_{B_{qp}^{2\mu-2/p-1}(\Omega)} \|c - \tilde{c}\|_{B_{qp}^{2\mu-2/p}(\Omega)} \Big) \\ &\quad \leq c_3 (\|w\|_{X_{\gamma,\mu}} + \|\tilde{w}\|_{X_{\gamma,\mu}}) \|w - \tilde{w}\|_{X_{\gamma,\mu}} \\ &\quad \leq c_4 R \|w - \tilde{w}\|_{X_{\gamma,\mu}}. \end{aligned}$$

This proves c). Hence, the proof is complete. □

Next, we want to apply Proposition 2.5.4 in order to show the existence of global solutions to (IVKS), provided the initial value is chosen close to an equilibrium. Therefore, we consider the set $\mathcal{E}_0 := \{(x, y) \subset \mathbb{R}^d \times \mathbb{R}^d : x = y\}$, which are equilibria of (IVKS). This set forms a d-dimensional subspace of $X_1 := D(\Delta_{n,w}) \times D(\Delta_N)$, hence a C^1-manifold with tangent space $\{(x, y) \subset \mathbb{R}^d \times \mathbb{R}^d : x = y\}$ at each point $(x_\star, y_\star) \in \mathcal{E}_0$.

Since $\mathcal{A}$ is a linear operator, we have $A_0 = \mathcal{A}$, which has the property of maximal L^p-regularity as described in the proof of Theorem 4.2.1. Furthermore, for $-\Delta_N + 1$ the spectrum $\sigma(-\Delta_N + 1)$ consists only of positive eigenvalues and it is $0 \notin \sigma(-\Delta_N + 1)$. Next, $-\Delta_{N,w}$ has 0 as an eigenvalue. Due to Proposition 2.4.7 this eigenvalue is semi-simple and the remaining part of $\sigma(-\Delta_{N,w})$ consists of positive eigenvalues. Thus, for some $\delta > 0$ we have $\sigma(-A_0) \setminus \{0\} \subset [\delta, \infty)$. The kernel of A_0 is given by

$$N(A_0) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\};$$

which coincides with the tangent space.

Then, by Proposition 2.5.4 we obtain thee following result on stability for the equilibria of (IVKS).

Theorem 4.2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain and assume that* (4.16) *is satisfied for $\mu = 1$.*

Then every equilibrium $w_\star \subset \mathcal{E}_0$ is stable in X_γ, i.e., there exists an $\varepsilon > 0$ such that the solution $w(t)$ of (IVKS) *with initial value $w_0 \in X_\gamma$ satisfying $\|w_0 - w_\star\|_{X_\gamma} \le \varepsilon$ exists globally in time and converges at an exponential rate in X_γ to some $w_\infty \in \mathcal{E}_0$ as $t \to \infty$.*

Bidomain Equations

CHAPTER 5

Time-Periodic Solutions to the Bidomain Equations

In this chapter, we investigate the bidomain system, a well established system describing the electrical wave propagation in the heart. Provided the system is innervated by periodic stimulation currents, we prove the existence of periodic solutions to this system from two different points of view.

The first approach is based on a periodic version of the classical theorem of Da Prato and Grisvard, which gives us the unique existence of periodic solutions to the linearized problem in real interpolation spaces. Then, by applying the contraction mapping principle we obtain the existence and uniqueness of a strong time-periodic solution to the bidomain equations under the assumption that the external forces are sufficiently small.

A second approach, which is comparable to the first one in the sense that we obtain the existence and uniqueness of strong time-periodic solutions for small external data, uses the semilinear version of the Arendt–Bu theorem instead of the periodic Da Prato–Grisvard theorem. Hence, we obtain solutions in the usual L^p-L^q-setting of maximal regularity.

In the third approach, which corresponds to the second point of view, we construct a weak periodic solution by using a Galerkin approximation and Brouwer's fixed point theorem. Then, using the global well-posedness result of Colli Franzone and Savaré [19] and a weak-strong uniqueness argument, we show the existence of a periodic solution to the bidomain system without assuming any smallness conditions on the external data. However, the periodic solution is not unique.

This chapter is structured as follows. First, we explain the system and give a review on the literature of the bidomain equations and recall some known results. In Section 5.2 we prove the existence and uniqueness of strong time-periodic solutions provided the external periodic stimulation currents satisfy a suitable smallness condition using two different techniques. Afterwards, in Section 5.3 we show that we can eliminate this smallness condition by paying the price of loosing uniqueness.

5.1 The Bidomain Equations

5.1.1 Explanation of the Model

The bidomain system is a well established system of equations describing the electrical activities of the heart. The system is given by

$$
\text{(BDE)} \quad \begin{cases} \partial_t u - \operatorname{div}(\sigma_i \nabla u_i) + f(u,w) = I_i & \text{in } (0,\infty)\times\Omega, \\ \partial_t u + \operatorname{div}(\sigma_e \nabla u_e) + f(u,w) = -I_e & \text{in } (0,\infty)\times\Omega, \\ \partial_t w + g(u,w) = 0 & \text{in } (0,\infty)\times\Omega, \\ u = u_i - u_e & \text{in } (0,\infty)\times\Omega, \\ \sigma_i \nabla u_i \cdot \nu = 0,\ \sigma_e \nabla u_e \cdot \nu = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0) = u_0,\ w(0) = w_0 & \text{in } \Omega. \end{cases}
$$

Here, the physical domain $\Omega \subset \mathbb{R}^d$ describes the myocardium and the outward unit normal vector to $\partial\Omega$ is denoted by ν. The unknown functions u_i and u_e model the intra- and extracellular electric potentials, and u denotes the transmembrane potential. The variable w, the so-called gating variable, corresponds to the ionic transport through the cell membrane. The anisotropic properties of the intra- and extracellular tissue parts are described by the conductivity matrices $\sigma_i(x)$ and $\sigma_e(x)$, whereas $I_i(t,x)$ and $I_e(t,x)$ denote the intra- and extracellular stimulation current, respectively.

The ionic transport is described by the nonlinear terms f and g. Starting with the pioneering work from Hodgkin and Huxley in the 1950s, there is a long tradition of mathematical models describing the propagation of impulses in electrophysiology. We refer to the survey article of Stevens [76] for more details. We consider various models for the ionic transport including the models by FitzHugh–Nagumo [27], Aliev–Panfilov [3], and Rogers–McCulloch [74]. The *FitzHugh–Nagumo model* reads as

$$
\begin{aligned} f(u,w) &= u(u-a)(u-1) + w = u^3 - (a+1)u^2 + au + w, \\ g(u,w) &= -\varepsilon(ku - w), \end{aligned}
$$

with $0 < a < 1$ and k, $\varepsilon > 0$.
In the *Rogers–McCulloch model* the functions f and g are given by

$$
\begin{aligned} f(u,w) &= bu(u-a)(u-1) + uw = bu^3 - b(a+1)u^2 + bau + uw, \\ g(u,w) &= -\varepsilon(ku - w), \end{aligned}
$$

with $0 < a < 1$ and b, k, $\varepsilon > 0$.
For the *Aliev–Panfilov model* we have

$$
\begin{aligned}
f(u,w) &= ku(u-a)(u-1) + uw = bu^3 - b(a+1)u^2 + bau + uw,\\
g(u,w) &= \varepsilon(ku(u-1-d) + w)
\end{aligned}
$$

with $0 < a, d < 1$ and b, k, $\varepsilon > 0$.

On a microscopic level, the cardiac cellular structure is described by two disjoint domains Ω_i and Ω_e, which denote the intra- and extracellular space, respectively, and which are separated by the the active membrane $\overline{\Gamma} = \partial\Omega_i \cap \partial\Omega_e$. The intra- and extracellular quantities are defined on the corresponding domains and the transmembrane potential u is a function on $\overline{\Gamma}$. After a homogenization procedure, see, e.g., [18,19], the macroscopic model of the bidomain equations is obtained. Here all membrane, intra-, and extracellular quantities are defined everywhere on Ω. The behavior of the ionic current through the cell membrane, described by the variable w, is coupled with the transmembrane voltage u by the equation in the third line of (BDE).

Introduced by Tung [78] in 1978 the bidomain model became an important model in cardiac electrophysiology over the last decades. Nevertheless, the rigorous mathematical analysis did not start until the work of Colli Franzone and Savaré [19] in 2002. They introduced a variational formulation of the bidomain problem and proved the existence and uniqueness of weak and strong solutions to the bidomain equations with FitzHugh–Nagumo type nonlinearities. A slightly more detailed review of their results is given in Section 5.3. Their results were extended by Veneroni [80] to more general ionic models including the Luo and Rudy I model.

Bourgault, Cordière, and Pierre [13] gave new input by pesenting a different approach to the bidomain system in 2009. Introducing a non-local, non-negative, and selfadjoint operator, the so-called bidomain operator, within the L^2-setting, they transformed the bidomain system into an abstract evolution equation and showed the existence and uniqueness of a local strong solution as well as the existence of a global weak solution for a large class of ionic models including the FitzHugh–Nagumo, the Aliev–Panfilov, and the Rogers–McCulloch models introduced above. For the weak solutions Kunisch and Wagner [56] later showed uniqueness and further regularity under some additional assumptions.

Giga and Kajiwara [33] extended the bidomain operator to the L^p-setting. They showed that the negative of the bidomain operator is the generator of a bounded analytic semigroup on $L^p(\Omega)$ for $p \in (1, \infty]$. Furthermore, they proved the existence and uniqueness of a local, strong solution to the bidomain system in this setting.

A further extension to these results was given by Hieber and Prüss. They proved maximal L^p-regularity for the bidomain operator in [39]. Furthermore, in [42] they proved global well-posedness for the bidomain equations. Therein they considered the case $I_i = I_e = 0$ with FitzHugh-Nagumo type non-linearities as well as stability of homogeneous equilibria.

A series of papers by Kunisch and Wagner [55–58] considers the optimal control problem for the bidomain system. For results concerning the bidomain equations with stochastic forcing modeled by a cylindrical Wiener process we refer to [9] and [37].

Most of the results mainly concern the well-posedness of the bidomain equations and results on the dynamics of the solution are even more rare. We refer here to the work of Mori and Matano [67], who studied for the first time the stability of front solutions of the bidomain equations.

In this context and since the bidomain model describes electrical activities in the heart, it is now a very natural question to ask, whether the bidomain equations admit time-periodic solutions. Therefore, consider the situation where the bidomain model is innervated by periodic intra- and extracellular stimulation currents I_i and I_e. Then, it is our goal to show that the innervated system admits a *strong* time-periodic solution of period T provided the outer forces I_i and I_e are both time-periodic of period $T > 0$.

In order to do so, we use three different approaches which will be described in detail in Sections 5.2 and 5.3. The first approach in Section 5.2 is based on a periodic version of the Da Prato–Grisvard theorem combined with the contraction mapping principle. A second approach in Section 5.2 uses the theory of maximal periodic regularity due to Arendt and Bu. Both approaches yield the existence of a *unique* strong time-periodic solution assuming that the external forces satisfy a suitable *smallness condition.* In Section 5.3, first the existence of a *weak* time-periodic solution is shown and then a *weak-strong uniqueness* argument is applied to show the existence of a strong time-periodic solution, this time without assuming smallness of the external currents but loosing uniqueness of the solution.

5.1.2 The Bidomain Operator

This subsection is devoted to fix some notation concerning the bidomain equations and formally introduce the bidomain operator in a weak as well as in a strong setting. Then, we collect some results concerning the bidomain operator and the corresponding bilinear form. Finally, the bidomain operator is used to rewrite the bidomain equations as an abstract evolution equation.

In the following, let the space dimension $d \geq 2$ be fixed and let $\Omega \subset \mathbb{R}^d$ denote a bounded domain with boundary $\partial\Omega$ of class C^2. For convenience, we introduce the following notation for the function spaces which we will use in the following and especially in Section 5.3

$$V = H^1(\Omega),\ H = L^2(\Omega),\ V' = (H^1(\Omega))'.$$

The spaces are equipped with their usual norms. Note that they are Hilbert spaces and that the canonical pairing of V' and V is denoted by ${}_{V'}\langle \cdot, \cdot \rangle_V$.

For the conductivity matrices σ_i and σ_e we make the following assumptions.

Assumption 5.1.1. *The conductivity matrices $\sigma_i, \sigma_e : \overline{\Omega} \to \mathbb{R}^{d\times d}$ are symmetric matrices and are functions of class $C^1(\overline{\Omega})$. Ellipticity is imposed by means of the following condition: there exist constants $\underline{\sigma}$, $\overline{\sigma}$ with $0 < \underline{\sigma} < \overline{\sigma}$ such that*

$$\underline{\sigma}|\xi|^2 \leq \langle \sigma_i(x)\xi, \xi \rangle \leq \overline{\sigma}|\xi|^2 \quad \textit{and} \quad \underline{\sigma}|\xi|^2 \leq \langle \sigma_e(x)\xi, \xi \rangle \leq \overline{\sigma}|\xi|^2 \tag{5.1}$$

for all $x \in \overline{\Omega}$ and all $\xi \in \mathbb{R}^d$. Moreover, it is assumed that

$$\begin{aligned} \sigma_i \nabla u_i \cdot \nu = 0 \quad &\Leftrightarrow \quad \nabla u_i \cdot \nu = 0 \quad \text{on } \partial\Omega, \\ \sigma_e \nabla u_e \cdot \nu = 0 \quad &\Leftrightarrow \quad \nabla u_e \cdot \nu = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{5.2}$$

According to [16] it is known that (5.2) is a biological reasonable assumption.

First, we want to introduce the bidomain operator in a weak setting as well as the corresponding bidomain bilinear form. In order to do so, denote by $V_{av}(\Omega) := \{u \in V : \int_\Omega u \,\mathrm{d}x = 0\}$ the set of functions in V with mean zero. Then, following [13], we define the bilinear forms

$$a_i(u,v) := \int_\Omega \sigma_i \nabla u \cdot \nabla v \,\mathrm{d}x, \quad a_e(u,v) := \int_\Omega \sigma_e \nabla u \cdot \nabla v \,\mathrm{d}x$$

for all $(u,v) \in V_{av} \times V_{av}$. Thanks to the uniform ellipticity condition (5.1) these bilinear forms are symmetric, continuous and uniformly elliptic on $V_{av} \times V_{av}$. Next, we use these bilinear forms to define the weak operators A_i and A_e from V_{av} onto V'_{av} by

$$\langle A_i u, v\rangle := a_i(u,v), \quad \langle A_e u, v\rangle := a_e(u,v)$$

for all $(u,v) \in V_{av} \times V_{av}$. Denoting by P_{av} the orthogonal projection from V to $V_{av}(\Omega)$, i.e., $P_{av}u := u - \frac{1}{|\Omega|}\int_\Omega u \,\mathrm{d}x$ and its transpose by $P_{av}^T : V'_{av} \to V'$, we collected everything necessary for the definition of the weak bidomain operator and the corresponding bidomain bilinear. To be more precise, they are given by

$$\begin{aligned} A &= P_{av}^T A_i (A_i + A_e)^{-1} A_e P_{av}, \\ a(u,v) &= \langle Au, v\rangle \end{aligned}$$

for all $(u,v) \in V \times V$. For the bidomain bilinear form we have the following result.

Lemma 5.1.2 ([13, Theorem 6]). *The bidomain bilinear form $a(\cdot,\cdot)$ is symmetric, continuous and coercive on V,*

$$\begin{aligned} \alpha\|u\|_V^2 &\le a(u,u) + \alpha\|u\|_H^2, && \textit{for all } u \in V, \\ |a(u,v)| &\le M\|u\|_V\|v\|_V, && \textit{for all } u, v \in V, \end{aligned}$$

for some constants α, $M > 0$. Furthermore, there exists an increasing sequence $0 = \lambda_0 < \cdots \le \lambda_i \le \cdots$ in $\mathbb{R}$ and an orthonormal Hilbert basis of H of eigenvectors $(\psi_i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $\psi_i \in V$, and $v \in V$ it is $a(\psi_i, v) = \lambda_i(\psi_i, v)$.

Now, we switch to the strong setting. Herein, we want to define the strong bidomain operator in the L^q-setting for $1 < q < \infty$. We will use the same notation as above for the weak setting since it will be clear from the context whether we consider the weak or strong formulation. Recall the space $L^q_{av}(\Omega) = \{u \in L^q(\Omega) : \int_\Omega u \, \mathrm{d}x = 0\}$ and denote by P_{av} the orthogonal projection from $L^q(\Omega)$ onto $L^q_{av}(\Omega)$, i.e., $P_{av}u := u - \frac{1}{|\Omega|}\int_\Omega u \, \mathrm{d}x$. We then define the elliptic operators A_i and A_e by

$$A_{i,e}u := -\nabla \cdot (\sigma_{i,e}\nabla u),$$
$$D(A_{i,e}) := \Big\{u \in W^{2,q}(\Omega) \cap L^q_{av}(\Omega) : \sigma_{i,e}\nabla u \cdot \nu = 0 \text{ a.e. on } \partial\Omega\Big\} \subset L^q_{av}(\Omega),$$

where $A_{i,e}$ and $\sigma_{i,e}$ indicates that either A_i and σ_i or A_e and σ_e are considered. Condition (5.2) implies that the domains coincide, i.e., $D(A_i) = D(A_e)$. Hence, for the elliptic operators A_i and A_e it is possible to define the sum $A_i + A_e$ with domain $D(A_i) = D(A_e)$. Note that the inverse operator $(A_i + A_e)^{-1}$ on $L^q_{av}(\Omega)$ is a bounded linear operator.

Following [33] we define the bidomain operator in the strong setting as follows. Let σ_i and σ_e satisfy Assumption 5.1.1. Then the bidomain operator A is defined as

$$A = A_i(A_i + A_e)^{-1}A_e P_{av} \tag{5.3}$$

with domain

$$D(A) := \{u \in W^{2,q}(\Omega) : \nabla u \cdot \nu = 0 \text{ a.e. on } \partial\Omega\}.$$

Recall the definition of the sector $\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\mathrm{arg}\lambda| < \theta\}$ for $\theta \in (0, \pi]$. Due to Giga and Kajiwara [33] we have the following resolvent estimates.

Proposition 5.1.3 ([33, Theorem 4.7, Theorem 4.9])**.** *Let $1 < q < \infty$, Ω be a bounded C^2-domain and let σ_i and σ_e satisfy Assumption 5.1.1. Then, for $\lambda \in \Sigma_\pi$ and $f \in L^q(\Omega)$, the resolvent problem*

$$(\lambda + A)u = f \quad \text{in } \Omega \tag{5.4}$$

has a unique solution $u \in D(A)$. Moreover, for each $\varepsilon \in (0, \pi/2)$ there exists a constant $C > 0$ such that for all $\lambda \in \Sigma_{\pi-\varepsilon}$ and all $f \in L^q(\Omega)$ the unique solution $u \in D(A)$ satisfies

$$|\lambda|\|u\|_{L^q(\Omega)} + |\lambda|^{1/2}\|\nabla u\|_{L^q(\Omega)} + \|\nabla^2 u\|_{L^q(\Omega)} \le C\|f\|_{L^q(\Omega)}.$$

Remark 5.1.4. In particular, this implies that $-A$ is the generator of a bounded analytic semigroup e^{-tA} on $L^q(\Omega)$.

Finally, we want to use the bidomain operator A to rewrite the bidomain system (BDE) as an abstract evolution equation. To this end, we assume the conservation of currents, i.e.,

$$\int_\Omega (I_i(t) + I_e(t))\,\mathrm{d}x = 0, \quad t \geq 0 \tag{5.5}$$

as well as $\int_\Omega u_e \,\mathrm{d}x = 0$. Then, proceeding as in [13] or [33] the bidomain equations (BDE) is equivalent to an evolution equation of the form

$$\text{(ABDE)} \qquad \begin{cases} \partial_t u + Au + f(u,w) = I, & \text{in } (0,\infty), \\ \partial_t w + g(u,w) = 0, & \text{in } (0,\infty), \\ u(0) = u_0, & \\ w(0) = w_0, & \end{cases}$$

where the modified external force I is given by

$$I := I_i - A_i(A_i + A_e)^{-1}(I_i + I_e). \tag{5.6}$$

Using the relations

$$\begin{aligned} u_e &= (A_i + A_e)^{-1}\{(I_i + I_e) - A_i P_{av} u\}, \\ u_i &= u + u_e. \end{aligned}$$

the intra- and extracelullar electric potentials u_i and u_e can be regained from the transmembrane potential u.

5.2 Unique Strong Time-Periodic Solutions

In this section, we show the unique existence of strong time-periodic solutions to the bidomain equations. In order to do so, the bidomain system is first reformulated into a coupled system. In this coupled system a 2×2 operator matrix $\mathcal{A}$ involving the bidomain operator A in one of its components will represent the linear part of (BDE).

Given a Banach space X and a T-periodic function $f : \mathbb{R} \to X$ whose restriction to $(0, T)$ belongs to $L^p(0, T; X)$, we understand by a strong T-periodic solution to the bidomain equations with right-hand side $(f, 0)$ a T-periodic tupel $(u, w) \in L^p(0, T; X)$ satisfying $(u', w') \in L^p(0, T; X)$ and $\mathcal{A}(u, w) \in L^p(0, T; X)$. This means in particular that $\mathcal{A}$ admit the property of maximal L^p-regularity.

In order to obtain a T-periodic solution to (BDE) within this regularity class, in our main approach described in Subsections 5.2.1-5.2.3 we choose as underlying Banach space the real interpolation space $D_A(\theta, p)$ for $\theta \in (0, 1)$, $1 \leq p < \infty$, where A still denotes the bidomain operator. This approach to T-periodic solutions for the linearized equation is then based on a periodic version of the classical Da Prato–Grisvard theorem, which we develop in Subsection 5.2.1. Note that we provide a rather long but self-contained proof for this periodic version which can be heavily shortened by applying results due to Da Prato and Grisvard [20]. Having this at hand, we apply then the contraction mapping principle in the space of maximal regularity to find a strong T-periodic solution of the nonlinear problem in a neighborhood of stable equilibrium points.

An alternative approach by the semilinear version of the Arendt–Bu theorem (Proposition 2.5.5), where the underlying Banach space X is a usual $L^q(\Omega)$, is presented in Subsection 5.2.4.

This section is structured as follows. In Subsection 5.2.1 we fix some notation and prove a periodic version of the Da Prato–Grisvard theorem. This will be extended in Subsection 5.2.2 to the semilinear setting. Then, in Subsection 5.2.3 the previous results will be applied to to the bidomain equations subject to various models for the ionic transport. Finally, in Subsection 5.2.4 we instead apply the semilinear Arendt–Bu theorem to the different bidomain models.

5.2.1 A Periodic Version of the Da Prato–Grisvard Theorem

Our main results on the unique existence of strong T-periodic solutions to (ABDE) are formulated in the real interpolation space $D_A(\theta, p)$ between $D(A)$ and the underlying space $L^q(\Omega)$. This choice of spaces is motivated by our aim to prove the existence and uniqueness of T-periodic solutions to the bidomain equations in a *strong* sense, where we have the possibility to consider the end-point case $p = 1$ in time. The classical Da Prato–Grisvard theorem ensures the maximal L^p-regularity for parabolic evolution equations in these spaces and our approach is based on a *periodic version of the Da Prato–Grisvard theorem.*

Before we prove this periodic version, let us first introduce the real interpolation space mentioned above and fix some notation which will be used in the rest of this section.

More specifically, let X be a Banach space and $-\mathcal{A}$ be the generator of a bounded analytic semigroup $\mathrm{e}^{-t\mathcal{A}}$ on X with domain $D(\mathcal{A})$. For $\theta \in (0,1)$ and $1 \le p < \infty$, we denote by $D_{\mathcal{A}}(\theta,p)$ the space defined as

(5.7)
$$D_{\mathcal{A}}(\theta,p) := \Big\{ x \in X : [x]_{\theta,p} := \Big(\int_0^\infty \|t^{1-\theta}\mathcal{A}\mathrm{e}^{-t\mathcal{A}}x\|_X^p \frac{\mathrm{d}t}{t} \Big)^{1/p} < \infty \Big\}.$$

Endowed with the norm $\|x\|_{\theta,p} := \|x\| + [x]_{\theta,p}$, this is a Banach space. For details and more on interpolation spaces we refer, e.g., to [64, 65]. Furthermore, it is well-known that $D_{\mathcal{A}}(\theta,p)$ coincides with the real interpolation space $(X, D(\mathcal{A}))_{\theta,p}$ and that the respective norms are equivalent. Due to [35, Corollary 6.5.5] the real interpolation space norm is equivalent to the homogeneous norm $[\cdot]_{\theta,p}$ if $\mathcal{A}$ is invertible. Consider in particular the bidomain operator A in $X = L^q(\Omega)$ for $1 < q < \infty$. Then, following Amann [5, Theorem 5.2], the space $(X, D(A))_{\theta,p}$ can be characterized as

$$(L^q(\Omega), D(A))_{\theta,p} = B^{2\theta}_{q,p}(\Omega), \quad 1 \le p \le \infty, \tag{5.8}$$

provided $2\theta \in (0, 1+1/q)$. Here $B^s_{q,p}(\Omega)$ denotes, as usual, the Besov space of order $s \ge 0$.

Let $0 < T < \infty$. We define the solution space $\mathbb{E}^{\mathrm{per}}_{\mathcal{A}}$ as

$$\begin{aligned} \mathbb{E}^{\mathrm{per}}_{\mathcal{A}} := \{ u \in W^{1,p}(0,T; D_{\mathcal{A}}(\theta,p)) : & \mathcal{A}u \in L^p(0,T; D_{\mathcal{A}}(\theta,p)) \\ & \text{and } u(0) = u(T) \} \end{aligned}$$

with norm

$$\|u\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}}} := \|u\|_{W^{1,p}(0,T;D_{\mathcal{A}}(\theta,p))} + \|\mathcal{A}u\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))}.$$

Then, the corresponding data space is defined by

$$\mathbb{F}_{\mathcal{A}} := L^p(0,T; D_{\mathcal{A}}(\theta,p)).$$

In our situation, where A denotes the bidomain operator, the solution space for the transmembrane potential u reads as

$$\begin{aligned}\mathbb{E}_A^{\mathrm{per}} = \{u \in W^{1,p}(0,T; D_A(\theta,p)) :& Au \in L^p(0,T; D_A(\theta,p)) \\ & \text{and } u(0) = u(T)\}.\end{aligned}$$

Moreover, for the gating variable w we define the solution space by

$$\mathbb{E}_w^{\mathrm{per}} := \{w \in W^{1,p}(0,T; D_A(\theta,p)) : w(0) = w(T)\}.$$

Combining the solution space for u and w, the solution space for the periodic bidomain system is defined as the product space

$$\mathbb{E} := \mathbb{E}_A^{\mathrm{per}} \times \mathbb{E}_w^{\mathrm{per}}.$$

Now, we can focus on the periodic version of Da Prato–Grisvard theorem. For $f \in L^p(0,T; D_{\mathcal{A}}(\theta,p))$ we consider

$$u(t) := \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}s, \qquad 0 < t < T. \tag{5.9}$$

Then, u is the unique mild solution to the abstract Cauchy problem

$$\begin{cases} u'(t) + \mathcal{A}u(t) = f(t), & 0 < t < T \\ u(0) = 0. \end{cases} \tag{ACP}$$

Due to the classical Da Prato and Grisvard theorem [20, Theorem 4.7] this solution satisfies the following maximal regularity estimate.

Proposition 5.2.1 ([20, Da Prato, Grisvard]). *Let $\theta \in (0,1)$, $1 \le p < \infty$, and $0 < T < \infty$. Then there exists a constant $C > 0$ such that for all $f \in L^p(0,T; D_{\mathcal{A}}(\theta,p))$, the function u given by* (5.9) *satisfies $u(t) \in D(\mathcal{A})$ for almost every $0 < t < T$ and*

$$\|\mathcal{A}u\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))} \le C\|f\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))}.$$

We remark at this point that the theorem above implies that the mild solution u to (ACP) is in fact a strong solution satisfying $u'(t)+\mathcal{A}u(t)=f(t)$ for almost every $0<t<T$.

Now we switch to the periodic setting. For $\theta\in(0,1)$, $1\le p<\infty$, and $0<T<\infty$ assume that $f:\mathbb{R}\to D_{\mathcal{A}}(\theta,p)$ is periodic of period T. Then the periodic version of (ACP) reads as

$$\text{(PACP)}\qquad \begin{cases} u'(t)+\mathcal{A}u(t)=f(t), & t\in\mathbb{R},\\ u(t)=u(t+T), & t\in\mathbb{R}.\end{cases}$$

Formally, a candidate for a solution u of (PACP) is given by

$$u(t):=\int_{-\infty}^{t}\mathrm{e}^{-(t-s)\mathcal{A}}f(s)\,\mathrm{d}s. \tag{5.10}$$

In the following lemma it is shown that this formal candidate is well-defined, periodic and continuous provided $\mathcal{A}$ and f fulfill certain conditions.

Lemma 5.2.2. *Let $f:\mathbb{R}\to D_{\mathcal{A}}(\theta,p)$ be a T-periodic function satisfying $f_{|(0,T)}\in L^p(0,T;D_{\mathcal{A}}(\theta,p))$ and assume that $0\in\rho(\mathcal{A})$. Then, the function u defined by* (5.10) *is well-defined, T-periodic, and satisfies $u\in C(\mathbb{R};D_{\mathcal{A}}(\theta,p))$.*

Proof. Let $k_0\in\mathbb{Z}$ be such that $-k_0T<t\le-(k_0-1)T$. Using Hölder's inequality, the periodicity of f, and the exponential decay of $\mathrm{e}^{-t\mathcal{A}}$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{t}\|\mathrm{e}^{-(t-s)\mathcal{A}}f(s)\|_{D_{\mathcal{A}}(\theta,p)}\,\mathrm{d}s\\
&\quad=\int_{-k_0T}^{t}\|\mathrm{e}^{-(t-s)\mathcal{A}}f(s)\|_{D_{\mathcal{A}}(\theta,p)}\,\mathrm{d}s+\sum_{k=k_0}^{\infty}\int_{-(k+1)T}^{-kT}\|\mathrm{e}^{-(t-s)\mathcal{A}}f(s)\|_{D_{\mathcal{A}}(\theta,p)}\,\mathrm{d}s\\
&\quad\le C\bigg(\int_0^{t+k_0T}\|f(s)\|^p_{D_{\mathcal{A}}(\theta,p)}\,\mathrm{d}s\bigg)^{\frac1p}\\
&\qquad+C\sum_{k=k_0}^{\infty}\mathrm{e}^{-\omega kT}\int_0^T\|\mathrm{e}^{-(T-s)\mathcal{A}}f(s)\|_{D_{\mathcal{A}}(\theta,p)}\,\mathrm{d}s\\
&\quad\le C\bigg(1+\sum_{k=k_0}^{\infty}\mathrm{e}^{-\omega kT}\bigg)\bigg(\int_0^T\|f(s)\|^p_{D_{\mathcal{A}}(\theta,p)}\,\mathrm{d}s\bigg)^{\frac1p}
\end{aligned}$$

for some $\omega > 0$. Hence, u is well-defined.

Next, we show that u is continuous. For $h > 0$ it is

$$\begin{aligned} &u(t+h) - u(t) \\ &\qquad = \int_t^{t+h} \mathrm{e}^{-(t+h-s)\mathcal{A}} f(s)\,\mathrm{d}s + \int_{-\infty}^{t} \mathrm{e}^{-(t-s)\mathcal{A}} [\mathrm{e}^{-h\mathcal{A}} - \mathrm{Id}] f(s)\,\mathrm{d}s. \end{aligned}$$

Due to the boundedness of the semigroup generated by $\mathcal{A}$ it is enough to consider the second integral of the formula. This integral resembles the expression from the proof of the well-posedness of u but with f being replaced by $[\mathrm{e}^{-h\mathcal{A}} - \mathrm{Id}]f$. Thus,

$$\begin{aligned} &\Big\| \int_{-\infty}^{t} \mathrm{e}^{-(t-s)\mathcal{A}} [\mathrm{e}^{-h\mathcal{A}} - \mathrm{Id}] f(s)\,\mathrm{d}s \Big\|_{D_{\mathcal{A}}(\theta,p)} \\ &\qquad \le C \bigg(\int_0^T \|[\mathrm{e}^{-h\mathcal{A}} - \mathrm{Id}] f(s)\|^p_{D_{\mathcal{A}}(\theta,p)}\,\mathrm{d}s \bigg)^{\frac{1}{p}} \end{aligned}$$

and the right-hand side tends to zero as $h \to 0$ by Lebesgue's theorem.

For the periodicity of u we use the transformation $s' = s + T$ and the periodicity of f. We obtain

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}s = \int_{-\infty}^{t+T} \mathrm{e}^{-(t-(s-T))\mathcal{A}} f(s-T)\,\mathrm{d}s \\ &= \int_{-\infty}^{t+T} \mathrm{e}^{-(t+T-s)\mathcal{A}} f(s)\,\mathrm{d}s = u(t+T). \end{aligned}$$

This finishes the proof. □

After this preparatory lemma, we can state the periodic version of the Da Prato–Grisvard theorem.

Theorem 5.2.3. *Let X be a Banach space and $-\mathcal{A}$ be the generator of a bounded analytic semigroup on X with $0 \in \rho(\mathcal{A})$. Let $\theta \in (0,1)$, $1 \le p < \infty$, and $0 < T < \infty$.*

Then there exists a constant $C > 0$ such that for all T-periodic functions $f : \mathbb{R} \to D_{\mathcal{A}}(\theta,p)$ with $f_{|(0,T)} \in L^p(0,T;D_{\mathcal{A}}(\theta,p))$ the function u defined by (5.10) lies in $C(\mathbb{R};D_{\mathcal{A}}(\theta,p))$, is periodic of period T, satisfies $u(t) \in D(\mathcal{A})$ for almost every $t \in \mathbb{R}$, and satisfies

$$\|\mathcal{A}u\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))} \le C \|f\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))}.$$

Proof. The continuity and periodicity of u follow with Lemma 5.2.2. Let $t \in [0,T)$ and use the transformation $s' = s + (k+1)T$ for $k \in \mathbb{N}_0$ as well as that f is periodic to write

$$\begin{aligned} u(t) &= \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}s + \sum_{k=0}^{\infty} \int_{-(k+1)T}^{-kT} \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}s \\ &= \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}s + \sum_{k=0}^{\infty} \mathrm{e}^{-(t+kT)\mathcal{A}} \int_0^T \mathrm{e}^{-(T-s)\mathcal{A}} f(s)\,\mathrm{d}s. \end{aligned} \tag{5.11}$$

In the following, use the notation

$$\mathfrak{u} := \int_0^T \mathrm{e}^{-(T-s)\mathcal{A}} f(s)\,\mathrm{d}s.$$

The theorem of Da Prato and Grisvard (Proposition 5.2.1) implies

$$\int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}s \in D(\mathcal{A}) \qquad (\text{a.e. } t \in (0,T))$$

and

$$\left\| t \mapsto \mathcal{A} \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}s \right\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))} \le C \|f\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))}.$$

Hence, using the exponential decay of the semigroup, it remains to prove the estimate

$$\|t \mapsto \mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))} \le C\|f\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))}. \tag{5.12}$$

In the following three steps, the estimate is proved for the seminorm $[\cdot]_{\theta,p}$ and in the final fourth step, it is shown for the ground space norm.

Step 1.

Let $\gamma_1, \gamma_2 \in (0,1)$ with $\gamma_1 + \gamma_2 = 1$ and $1/p' < \gamma_2 < 1 - \theta + 1/p'$, where p' denotes the Hölder conjugate exponent to p. Then, the boundedness and the analyticity of the semigroup, followed by the linear transformation

$s' = T + t - s$ and Hölder's inequality imply

$$
\begin{aligned}
&\|\mathcal{A}e^{-\tau\mathcal{A}}\mathcal{A}e^{-t\mathcal{A}}\mathfrak{u}\|_X \\
&\quad \le C\int_0^T \frac{1}{(T+\tau+t-s)^{\gamma_1}}\frac{1}{(T+\tau+t-s)^{\gamma_2}}\|\mathcal{A}e^{-(T+\tau+t-s)/2\mathcal{A}}f(s)\|_X \,\mathrm{d}s \\
&\quad = C\int_t^{T+t} \frac{1}{(\tau+s)^{\gamma_1}}\frac{1}{(\tau+s)^{\gamma_2}}\|\mathcal{A}e^{-(\tau+s)/2\mathcal{A}}f(T+t-s)\|_X \,\mathrm{d}s \\
&\quad \le C(\tau+t)^{1/p'-\gamma_2}\left(\int_t^{T+t}\frac{1}{(\tau+s)^{\gamma_1 p}}\|\mathcal{A}e^{-(\tau+s)/2\mathcal{A}}f(T+t-s)\|_X^p \,\mathrm{d}s\right)^{\frac{1}{p}}.
\end{aligned}
$$

Note that $1/p' < \gamma_2$ was eminent in the calculation above. Next, $t > 0$ implies

$$
\begin{aligned}
&\|\mathcal{A}e^{-\tau\mathcal{A}}\mathcal{A}e^{-t\mathcal{A}}\mathfrak{u}\|_X \\
&\quad \le C\tau^{1/p'-\gamma_2}\left(\int_t^{T+t}\frac{1}{(\tau+s)^{\gamma_1 p}}\|\mathcal{A}e^{-(\tau+s)/2\mathcal{A}}f(T+t-s)\|_X^p \,\mathrm{d}s\right)^{\frac{1}{p}}.
\end{aligned}
\tag{5.13}
$$

Step 2.

An application of (5.13) and Fubini's theorem yields

$$
\begin{aligned}
&\int_0^T \|\mathcal{A}e^{-\tau\mathcal{A}}\mathcal{A}e^{-t\mathcal{A}}\mathfrak{u}\|_X^p \,\mathrm{d}t \\
&\quad \le C\tau^{p(1/p'-\gamma_2)} \\
&\qquad \cdot \int_0^{2T}\int_{\max\{0,s-T\}}^{\min\{T,s\}} \frac{1}{(\tau+s)^{\gamma_1 p}}\|\mathcal{A}e^{-(\tau+s)/2\mathcal{A}}f(T+t-s)\|_X^p \,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}
$$

Note that the inner integral can be estimated by using $\min\{T,s\} \le s$. The transformation $t' = T + t - s$ delivers then the estimate

$$
\begin{aligned}
&\|t \mapsto \mathcal{A}e^{-\tau\mathcal{A}}\mathcal{A}e^{-t\mathcal{A}}\mathfrak{u}\|_{\mathrm{L}^p(0,T;X)}^p \\
&\quad \le C\tau^{p(1/p'-\gamma_2)}\int_0^{2T}\int_{\max\{0,T-s\}}^{T}\frac{1}{(\tau+s)^{\gamma_1 p}}\|\mathcal{A}e^{-(\tau+s)/2\mathcal{A}}f(t)\|_X^p \,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}
\tag{5.14}
$$

Step 3.

Use Fubini's theorem first and then (5.14) to estimate the full norm by

$$\int_0^T [\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}]_{\theta,p}^p \,\mathrm{d}t \\ \leq C\int_0^\infty \tau^{\gamma-1}\int_0^{2T}\int_{\max\{0,T-s\}}^T \frac{1}{(\tau+s)^{\gamma_1 p}}\|\mathcal{A}\mathrm{e}^{-(\tau+s)/2\mathcal{A}}f(t)\|_X^p \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}\tau,$$

where $\gamma = p(1+1/p'-\theta-\gamma_2)$. Apply Fubini's theorem followed by the substitution $s' = \tau + s$ to get

$$\int_0^T [\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}]_{\theta,p}^p \,\mathrm{d}t \leq C\int_0^T\int_0^\infty \tau^{\gamma-1}\int_{T+\tau-t}^{2T+\tau}\frac{1}{s^{\gamma_1 p}}\|\mathcal{A}\mathrm{e}^{-s/2\mathcal{A}}f(t)\|_X^p \,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}t.$$

Finally, use Fubini's theorem in order to calculate the τ-integral (here $\gamma_2 < 1-\theta+1/p'$ is essential) and note that $t-T$ is negative and γ positive to get

$$\begin{aligned}\int_0^T [\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}]_{\theta,p}^p \,\mathrm{d}t &\leq \frac{C}{\gamma}\int_0^T\int_{T-t}^\infty \frac{1}{s^{\gamma_1 p}}\|\mathcal{A}\mathrm{e}^{-s/2\mathcal{A}}f(t)\|_X^p(s+t-T)^\gamma \,\mathrm{d}s\,\mathrm{d}t \\ &\leq \frac{C}{\gamma}\int_0^T\int_{T-t}^\infty s^{\gamma-\gamma_1 p}\|\mathcal{A}\mathrm{e}^{-s/2\mathcal{A}}f(t)\|_X^p \,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

The proof is concluded by the definition of γ and of the real interpolation space norm, since this gives

$$\int_0^T [\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}]_{\theta,p}^p \,\mathrm{d}t \leq \frac{2^{p(1-\theta)}C}{2\gamma}\|f\|_{L^p(0,T;D_\mathcal{A}(\theta,p))}^p.$$

Step 4.

In this step, the term $\int_0^T \|\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}\|_X \,\mathrm{d}t$ is estimated. It is known, see [35, Corollary 6.6.3], that $D_\mathcal{A}(\vartheta,1) \hookrightarrow D(\mathcal{A}^\vartheta)$ and that $D_\mathcal{A}(\theta,p) \hookrightarrow D_\mathcal{A}(\vartheta,1)$ for every $0<\vartheta<\theta$. Thus,

$$D_\mathcal{A}(\theta,p) \hookrightarrow D(\mathcal{A}^\vartheta).$$

Now, let $\vartheta_1,\vartheta_2,\vartheta_3 \in (0,1)$ with $\vartheta_1+\vartheta_2+\vartheta_3 = 1$, $\vartheta_1 < \theta$, $\vartheta_2 p' < 1$ and $\vartheta_3 p < 1$, where p' denotes the Hölder conjugate exponent to p. Then, the bounded analyticity of $e^{-t\mathcal{A}}$, Hölder's inequality and the embedding above imply

$$\begin{aligned}
\|\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}\|_X &= \|\mathcal{A}^{\vartheta_3}\mathrm{e}^{-t\mathcal{A}}\int_0^T \mathcal{A}^{\vartheta_2}\mathrm{e}^{-(T-s)\mathcal{A}}\mathcal{A}^{\vartheta_1}f(s)\,\mathrm{d}s\|_X \\
&\le Ct^{-\vartheta_3}\int_0^T (T-s)^{-\vartheta_2}\|A^{\vartheta_1}f(s)\|_X\,\mathrm{d}s \\
&\le Ct^{-\vartheta_3}\left(\int_0^T (T-s)^{-\vartheta_2 p'}\,\mathrm{d}s\right)^{\frac{1}{p'}}\left(\int_0^T \|A^{\vartheta_1}f(s)\|_X^p\,\mathrm{d}s\right)^{\frac{1}{p}} \\
&\le Ct^{-\vartheta_3}\|f\|_{L^p(0,T:D_{\mathcal{A}}(\theta,p))}.
\end{aligned}$$

Consequently,

$$\int_0^T \|\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}\|_X\,\mathrm{d}t \le C\|f\|_{L^p(0,T:D_{\mathcal{A}}(\theta,p))}. \qquad \square$$

Remark 5.2.4. The proof of the key estimate (5.12) provided above is self-contained using only elementary ingredients as Fubini's theorem and Hölder's inequality as well as the properties of $\mathcal{A}$. Note that the proof can be greatly shortened by using more involved results from [20].

To be more precise, let $\mathcal{T}$ be the space of traces at 0 of functions belonging to the maximal regularity class $MR := \{u \in L^p(0,T;D(\mathcal{A})) : u', \mathcal{A}u \in L^p(0,T;D_{\mathcal{A}}(\theta,p))\}$. Following Lemma 4.14 and Theorem 4.15 of [20], estimate (5.12) holds true provided $\mathfrak{u}$ belongs to this space of traces $\mathcal{T}$. Observe that the function $v : [0,T] \to X, t \mapsto \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}}f(s)\mathrm{d}s$ belongs to MR. Thus, the same is true for $v(T-\cdot)$. Consequently, $\mathfrak{u} = v(T) \in \mathcal{T}$ which proves estimate (5.12).

It remains to verify that u defined by (5.10) is actually the unique strong time-periodic solution to (PACP) under the assumptions of Theorem 5.2.3.

Proposition 5.2.5. *Under the hypotheses of Theorem 5.2.3 the function u defined by* (5.10) *is the unique strong solution to* (PACP)*, i.e., u is the unique periodic function of period T in $C(\mathbb{R};X)$ that is for almost every $t \in \mathbb{R}$ differentiable in t, satisfies $u(t) \in D(\mathcal{A})$, and $\mathcal{A}u \in L^p(0,T;X)$, and u solves*

$$u'(t) + \mathcal{A}u(t) = f(t). \tag{5.15}$$

Proof. By Lemma 5.2.2 u is periodic. Since $D_{\mathcal{A}}(\theta, p)$ continuously embeds into X the same lemma implies $u \in C(\mathbb{R}; X)$.

Assume first that $f_{|(0,T)} \in L^p(0,T; D(\mathcal{A}))$. Then, by

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = \int_{-\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-(t-s)\mathcal{A}}f(s)\,\mathrm{d}s + f(t) = \int_{-\infty}^{t} -\mathcal{A}\mathrm{e}^{-(t-s)\mathcal{A}}f(s)\,\mathrm{d}s + f(t),$$

it follows that u defined by (5.10) is differentiable, satisfies $u(t) \in D(\mathcal{A})$, and solves (5.15) for every $t \in \mathbb{R}$. The density of $L^p(0,T; D(\mathcal{A}))$ in $L^p(0,T; D_{\mathcal{A}}(\theta, p))$ and the estimate proven in Theorem 5.2.3 imply that all these properties carry over to all right-hand sides in $L^p(0,T; D_{\mathcal{A}}(\theta, p))$ (but only for almost every $t \in \mathbb{R}$) by an approximation argument.

Finally, in order to show the uniqueness of the solution, assume that $v \in C(\mathbb{R}; X)$ with $v', \mathcal{A}v \in L^p(0,T; X)$ is another T-periodic function which satisfies the equation for almost every $t \in \mathbb{R}$. Define $w := u - v$. Then w satisfies

$$w'(t) = -\mathcal{A}w(t) \qquad (\text{a.e. } t \in \mathbb{R}).$$

In this case, for $t > 0$, w can be written by means of the semigroup as $w(t) = \mathrm{e}^{-t\mathcal{A}}(u(0) - v(0))$. Now, the exponential decay of the semigroup and the periodicity of w imply that w must be zero for all $t \in \mathbb{R}$. □

We close this subsection by giving the maximal regularity estimate for the solution u.

Remark 5.2.6. Combining Theorem 5.2.3 and Proposition 5.2.5 shows that for each periodic f with period T and $f_{|(0,T)} \in L^p(0,T; D_{\mathcal{A}}(\theta, p))$ also $u'_{|(0,T)} \in L^p(0,T; D_{\mathcal{A}}(\theta, p))$. The same is true for u since $0 \in \rho(\mathcal{A})$. Summarizing, there exists a constant $C > 0$ such that

$$\|u\|_{\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}} \leq C\|f\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))}. \tag{5.16}$$

5.2.2 Time-Periodic Solutions for Semilinear Equations

In this subsection, we extend the result from the previous subsection to the semilinear setting. To be more precise, we use the periodic version of the Da Prato–Grisvard theorem to solve the linear problem as described in detail in the last subsection. Then, by employing Banach's fixed point theorem we obtain the existence of a unique, strong time-periodic solution solution to semilinear parabolic evolution equations. The framework that is presented here includes all the models mentioned in Subsection 5.1.1.

In the following, let $-\mathcal{A}$ be the generator of a bounded analytic semigroup $\mathrm{e}^{-t\mathcal{A}}$ on a Banach space X with domain $D(\mathcal{A})$ and $0 \in \rho(\mathcal{A})$. For $T > 0$, $\theta \in (0,1)$, and $1 \leq p < \infty$ let $f : \mathbb{R} \to D_{\mathcal{A}}(\theta, p)$ be periodic of period T with $f_{|(0,T)} \in L^p(0,T; D_{\mathcal{A}}(\theta,p))$. It is our aim tho show the existence and uniqueness of strong time-periodic solutions of

$$\text{(NACP)} \qquad \begin{cases} u'(t) + \mathcal{A}u(t) = F(u(t)) + f(t) & t \in \mathbb{R} \\ u(t) = u(t+T) & t \in \mathbb{R} \end{cases}$$

assuming that the external force f is sufficiently small. The solution u will be constructed in the space of maximal regularity $\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}$ defined in the end of Subsection 5.2.1. Recall the corresponding data space

$$\mathbb{F}_{\mathcal{A}} = L^p(0,T; D_{\mathcal{A}}(\theta,p))$$

and for convenience let $B_\rho := B_{\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}}(0,\rho)$ for some $\rho > 0$. For the nonlinear term F, we assume the following properties.

Assumption 5.2.7. *There exists $R > 0$ such that the nonlinear term F is a mapping from B_R into $\mathbb{F}_{\mathcal{A}}$ and satisfies*

$$F \in C^1(B_R; \mathbb{F}_{\mathcal{A}}), \quad F(0) = 0, \quad \textit{and} \quad DF(0) = 0,$$

where $DF : B_R \to \mathcal{L}(\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}, \mathbb{F}_{\mathcal{A}})$ denotes the Fréchet derivative.

Using this assumption on the nonlinearity F, we now state the theorem on existence and uniqueness of solutions to (NACP) for small external forcings f.

Theorem 5.2.8. *Let $T > 0$, $\theta \in (0,1)$, $1 \leq p < \infty$, and F and $R > 0$ subject to Assumption 5.2.7. Then there are constants $r \leq R$ and $c = c(T,\theta,p,r) > 0$ such that if $f : \mathbb{R} \to D_{\mathcal{A}}(\theta,p)$ is T-periodic with $\|f\|_{\mathbb{F}_{\mathcal{A}}} \leq c$, then there exists a unique solution $u : \mathbb{R} \to D_{\mathcal{A}}(\theta,p)$ of* (NACP) *with the same period T and $u_{|(0,T)} \in B_r$.*

Proof. Let $S : B_R \to \mathbb{E}_{\mathcal{A}}^{\mathrm{per}}, v \mapsto u_v$ be the solution operator of the linear equation

$$u_v'(t) + \mathcal{A}u_v(t) = F(v(t)) + f(t) \quad \text{in } (0,T)$$

with $u_v(0) = u_v(T)$. This is well-defined since $F(v) \in \mathbb{F}_{\mathcal{A}}$ by Assumption 5.2.7. Thus, by Proposition 5.2.5 and Remark 5.2.6, the solution u_v uniquely exists and lies in $\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}$.

In order to apply the contraction mapping principle, we have to show that S is a self-mapping and a contraction. First, we prove that this solution operator maps B_r to B_r for some $r \leq R$. Let $M > 0$ denote the infimum of all constants C satisfying (5.16). Choose $r > 0$ small enough such that

$$\sup_{w \in B_r} \|DF(w)\|_{\mathcal{L}(\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}, \mathbb{F}_{\mathcal{A}})} \leq \frac{1}{2M},$$

which is possible by Assumption 5.2.7. Let f satisfy $\|f\|_{\mathbb{F}_{\mathcal{A}}} \leq r/(2M) =: c$. Then, by virtue of (5.16) as well as the mean value theorem, for any $v \in B_r$ we estimate

$$\begin{aligned}
\|S(v)\|_{\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}} &\leq M(\|F(v)\|_{\mathbb{F}_{\mathcal{A}}} + \|f\|_{\mathbb{F}_{\mathcal{A}}}) \\
&\leq M(\sup_{w \in B_r} \|DF[w]\|_{\mathcal{L}(\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}, \mathbb{F}_{\mathcal{A}})} \|v\|_{\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}} + \|f\|_{\mathbb{F}_{\mathcal{A}}}) \\
&\leq r.
\end{aligned}$$

So $S(B_r) \subset B_r$. Similarly, for any $v_1, v_2 \in B_r$ we obtain

$$\begin{aligned}
\|S(v_1) - S(v_2)\|_{\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}} &\leq M \sup_{w \in B_r} \|DF[w]\|_{\mathcal{L}(\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}, \mathbb{F}_{\mathcal{A}})} \|v_1 - v_2\|_{\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}} \\
&\leq \frac{1}{2} \|v_1 - v_2\|_{\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}}.
\end{aligned}$$

Consequently, the solution operator S is a contraction on B_r and the contraction mapping theorem is applicable. Then, the solution to (NACP) is defined as follows. Let u be the unique fixed point of S. Since $Su = u$, u satisfies $u(0) = u(T)$ and thus can be extended periodically to the whole real line. This function solves (NACP). $\square$

In the following we show that nonlinearities of the models introduced in Subsection 5.1.1 as well as those of the bidomain Allen–Cahn equation, which we introduce in detail in the next subsection, satisfy Assumption 5.2.7. Hence, in the models one of the following situations occurs:

- The bidomain operator A appears only in the first but not in the second equation of the bidomain models and the nonlinearity depends nonlinearly on the gating variable w. (FitzHugh–Nagumo, Aliev–Panfilov, Rogers–McCulloch)
- The ODE and the gating variable w are omitted. (Allen–Cahn)

This means, that, considering the first situation, the linear part of the bidomain models can be represented by an operator matrix whose first component of the domain embeds into a $W^{2,q}$-space. For the gating variable w we observe, that there appears no smoothing in the spatial variables since the dynamics of the gating variable is described only by an ODE in time. However, as it is our goal to apply Theorem 5.2.8 and as the nonlinearity of the first equation depends nonlinearly on w, at least in the models of Aliev–Panfilov and Rogers–McCulloch, w must be contained in $D_A(\theta, p)$. Otherwise one cannot view the nonlinearity as a suitable right-hand side as it is done in in the beginning of this subsection. This justifies the choice $D_A(\theta, p)$ as the ground space for the gating variable w as done in Subsection 5.2.1.

In order to translate this situation in the abstract setup, assume in the following, that $-\mathcal{A}$ is the generator of a bounded analytic semigroup on a Banach space $X = X_1 \times X_2$, with domain $D(\mathcal{A}) = D(A_1) \times D(A_2)$, and $0 \in \rho(\mathcal{A})$. For $1 < q < \infty$, $1 \leq p < \infty$, and $\theta \in (0, 1)$ these spaces are given by

$$X_1 = L^q(\Omega), \quad D(A_1) = D(A), \quad \text{and} \quad X_2 = D(A_2) = D_A(\theta, p).$$

Moreover, the nonlinearities are of one of the two following types. For $a_1, a_2, a_3, a_4 \in \mathbb{R}$ let

$$F_1(u_1, u_2) := \begin{pmatrix} a_1 u_1^2 + a_2 u_1^3 + a_3 u_1 u_2 \\ a_4 u_1^2 \end{pmatrix}$$

and for b_1, $b_2 \in \mathbb{R}$ let

$$F_2(u_1) := b_1 u_1^2 + b_2 u_1^3.$$

Note that the nonlinearities of the models of FitzHugh–Nagumo, Rogers–McCulloch, and Aliev–Panfilov fit in the first type F_1 whereas the one of the Allen–Cahn model is of type F_2.

The condition $0 \in \rho(\mathcal{A})$ seems inappropriate at first glance since for the bidomain operator A we have $0 \notin \rho(A)$. However, before applying Theorem 5.2.8 to the various bidomain models we will linearize the equations around suitable stable stationary solutions and for this linearized situation $0 \in \rho(\mathcal{A})$ will be true.

In the following, we concentrate only on F_1, since the results for F_2 may be proved in a similar way. We have to show that F_1 satisfies Assumption 5.2.7. Therefore, we need the following lemma.

Lemma 5.2.9. *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^2-domain. Let $s > 0$ and $p, q \in [1, \infty)$ with $s > d/q$ or, in the case $p = 1$, let $s \geq d/q$. Then $B^s_{q,p}(\Omega)$ is a Banach algebra.*

Proof. If $\Omega = \mathbb{R}^d$ and $s > d/q$ this readily follows by Sobolev's embedding combined with [8, Cor. 2.86]. If $s = d/q$ and $p = 1$, then [8, Cor. 2.86] has to be combined with [8, Prop. 2.39]. Finally, the algebra property on domains is directly transferred from the whole space since bounded C^2-domains are Sobolev extension domains. □

Having this lemma at hand, we elaborate the conditions on p, q, and θ which guarantee that F_1 maps $\mathbb{E}^{\text{per}}_{\mathcal{A}}$ into $\mathbb{F}_{\mathcal{A}}$.

Proposition 5.2.10. *Let $2\theta \in (0, 1)$, $1 \leq p < \infty$, and $1 < q < \infty$ satisfy $2\theta > d/q$ or, if $p = 1$ let $2\theta \geq d/q$. Then there exists a constant $C > 0$ such that*

$$\|F_1(u_1, u_2)\|_{\mathbb{F}_{\mathcal{A}}} \leq C\Big(\|u_1\|^2_{\mathbb{E}^{\text{per}}_{A_1}} + \|u_1\|^3_{\mathbb{E}^{\text{per}}_{A_1}} + \|u_1\|_{\mathbb{E}^{\text{per}}_{A_1}} \|u_2\|_{\mathbb{E}^{\text{per}}_{A_2}}\Big)$$

for all $u_1 \in \mathbb{E}^{\text{per}}_{A_1}$ and $u_2 \in \mathbb{E}^{\text{per}}_{A_2}$.

Proof. Let $\alpha \in \{1,2,3\}$ and $\beta \in \{0,1\}$. Recall that $D_A(\theta,p) = B^{2\theta}_{q,p}(\Omega)$. Lemma 5.2.9 implies that

$$\begin{aligned}&\|u_1^\alpha u_2^\beta\|_{L^p(0,T;D_A(\theta,p))}\\ &\quad\leq C\|u_1\|_{L^p(0,T;D_A(\theta,p))}\|u_1\|^{\alpha-1}_{L^\infty(0,T;D_A(\theta,p))}\|u_2\|^\beta_{L^\infty(0,T;D_A(\theta,p))}\end{aligned}$$

for some constant $C > 0$. Using the embedding

$$W^{1,p}(0,T;B^{2\theta}_{q,p}(\Omega)) \subset L^\infty(0,T;B^{2\theta}_{q,p}(\Omega))$$

yields

$$\|u_1^\alpha u_2^\beta\|_{L^p(0,T;D_A(\theta,p))} \leq C\|u_1\|^\alpha_{W^{1,p}(0,T;D_A(\theta,p))}\|u_2\|^\beta_{W^{1,p}(0,T;D_A(\theta,p))}.$$

By definition of $\mathbb{E}^{\mathrm{per}}_{A_1}$ and $\mathbb{E}^{\mathrm{per}}_{A_2}$ this proves the proposition. □

Finally, by definition of F_1 it follows that $F_1(0,0) = 0$. Moreover, due to the polynomial structure of F_1 it is clear that F_1 is Fréchet differentiable with $DF_1(0,0) = 0$. Hence, we have the following proposition.

Proposition 5.2.11. *With the definitions of this subsection the nonlinearities F_1 and F_2 satisfy Assumption 5.2.7.*

5.2.3 Periodic Solutions for Various Ionic Models

We apply the abstract results from the previous subsections to the bidomain equations subject to different types of ionic models as well as to the bidomain Allen–Cahn equation. Therefore, we remark that the linear part of the bidomain systems will be represented as an operator matrix and it will be eminent that the negative of this operator matrix generates a bounded analytic semigroup. This will be proven in the following lemma.

Lemma 5.2.12. *Let $-B$ be the generator of a bounded analytic semigroup on a Banach space X_1 with $0 \in \rho(B)$, $1 \leq p < \infty$, and $\theta \in (0,1)$. Let $X_2 = D_B(\theta,p)$ and define for $e > 0$ and $b,c \geq 0$ the operator $\mathcal{A} : X := X_1 \times X_2 \to X$ with domain $D(\mathcal{A}) := D(B) \times X_2$ by*

$$\mathcal{A} := \begin{pmatrix} B & b \\ -c & e \end{pmatrix}.$$

Then $-\mathcal{A}$ generates a bounded analytic semigroup on X with $0 \in \rho(\mathcal{A})$.

Proof. Let Σ_ω, $\omega \in (\pi/2, \pi]$, be a sector that satisfies $\rho(-B) \subset \Sigma_\omega$ with

$$\|\lambda(\lambda + B)^{-1}\|_{\mathcal{L}(X_1)} \leq C \qquad (\lambda \in \Sigma_\omega).$$

First note that $0 \in \rho(\mathcal{A})$; its inverse being

$$\mathcal{A}^{-1} = \begin{pmatrix} e & -b \\ c & B \end{pmatrix} (bc + eB)^{-1}.$$

Note that the choice $X_2 = D_B(\theta, p)$ is used here as $\mathcal{A}^{-1}$ is only an operator from $X_1 \times X_2$ onto $D(B) \times X_2$ if $D(B) \subset X_2 \subset X_1$ and if $B(bc + eB)^{-1}$ maps X_2 into X_2. By the definition of $D_B(\theta, p)$ in (5.7) this latter is satisfied.

For the resolvent problem let $\lambda \in \Sigma_\beta$, $\beta \in (\pi/2, \omega)$ to be chosen. Then,

$$(\lambda + \mathcal{A})^{-1} = (\lambda + e)^{-1} \begin{pmatrix} \lambda + e & -b \\ c & \lambda + B \end{pmatrix} \Big(\lambda + \frac{bc}{\lambda + e} + B\Big)^{-1}$$

whenever $\lambda + \frac{bc}{\lambda+e} \in \rho(-B)$. To determine the angle β for which $\lambda + \frac{bc}{\lambda+e} \in \rho(-B)$ distinguish between the cases $|\lambda| < M$ and $|\lambda| \geq M$ for some suitable constant $M > 0$. Note that only the case $b, c > 0$ is of interest. Let $C_\omega > 0$ be a constant depending solely on ω such that $|\lambda + e| \geq C_\omega(|\lambda| + e)$. Choose M such that $|\lambda| \geq M$ if and only if

$$C_\omega \sin(\omega - \beta)[|\lambda|^2 + e|\lambda|] \geq 2bc. \tag{5.17}$$

This implies

$$\Big|\frac{bc}{\lambda + e}\Big| \leq \frac{bc}{C_\omega(|\lambda| + e)} \leq \frac{|\lambda| \sin(\omega - \beta)}{2}$$

and thus that $\lambda + \frac{bc}{\lambda+e} \in \Sigma_\omega$. Moreover,

$$\Big|\lambda + \frac{bc}{\lambda + e}\Big| \geq |\lambda|\Big(1 - \frac{\sin(\omega - \beta)}{2}\Big). \tag{5.18}$$

Next, choose β close enough to $\pi/2$ such that

$$M\sin(\beta-\pi/2)\le \frac{bce}{bc+(e+M)^2}. \tag{5.19}$$

Notice that M itself depends on β, however, it depends only uniformly on its distance to ω by (5.17). In the case $|\lambda|<M$ the validity of (5.19) together with trigonometric considerations implies that $\operatorname{Re}\left(\lambda+\frac{bc}{\lambda+e}\right)\ge 0$ proving that under conditions (5.17) and (5.19) we have $\lambda+\frac{bc}{\lambda+e}\in\Sigma_\omega$ whenever $\lambda\in\Sigma_\beta$. We conclude that $\lambda\in\rho(-\mathcal{A})$. To obtain the resolvent estimate, we calculate

$$\begin{aligned}
\|\lambda(\lambda+\mathcal{A})^{-1}\|_{\mathcal{L}(X)} \le{}& \Big\|\lambda\Big(\lambda+\frac{bc}{\lambda+e}+B\Big)^{-1}\Big\|_{\mathcal{L}(X_1)}\\
&+\Big|\frac{\lambda b}{\lambda+e}\Big|\Big\|\Big(\lambda+\frac{bc}{\lambda+e}+B\Big)^{-1}\Big\|_{\mathcal{L}(X_2,X_1)}\\
&+\Big|\frac{\lambda c}{\lambda+e}\Big|\Big\|\Big(\lambda+\frac{bc}{\lambda+e}+B\Big)^{-1}\Big\|_{\mathcal{L}(X_1,X_2)}\\
&+\Big|\frac{\lambda}{\lambda+e}\Big|\Big\|(\lambda+B)\Big(\lambda+\frac{bc}{\lambda+e}+B\Big)^{-1}\Big\|_{\mathcal{L}(X_2)}.
\end{aligned}$$

The first term on the right-hand side is directly handled by the resolvent estimate of B. The second is treated by this resolvent estimate as well and by noting that $X_2\subset X_1$. The fourth term is estimated by using that the definition of X_2 in (5.7) implies resolvent estimates in X_2 (the resolvent commutes with the semigroup appearing in (5.7)). For the third term, the estimate follows from the invertibility of B and the interpolation inequality $\|x\|_{X_2}\le C\|x\|_{X_1}^{1-\theta}\|Bx\|_{X_1}^{\theta}$. Altogether, this yields

$$\begin{aligned}
&\|\lambda(\lambda+\mathcal{A})^{-1}\|_{\mathcal{L}(X)}\\
&\le C\bigg(|\lambda|+\Big|\frac{\lambda b}{\lambda+e}\Big|+\Big|\frac{\lambda c}{\lambda+e}\Big|\Big|\lambda+\frac{bc}{\lambda+e}\Big|^{\theta}+\Big|\frac{\lambda^2}{\lambda+e}\Big|\bigg)\Big|\lambda+\frac{bc}{\lambda+e}\Big|^{-1}\\
&\quad+C\Big|\frac{\lambda}{\lambda+e}\Big|.
\end{aligned}$$

The resolvent estimate for $|\lambda|\ge M$ follows by means of the uniform boundedness of the term $|\lambda/(\lambda+e)|$ and by (5.18).

For $|\lambda|<M$ the function $\lambda\mapsto\lambda(\lambda+\mathcal{A})^{-1}$ is continuous on $\overline{\Sigma_\beta}\cap\overline{B(0,M)}$ since $0\in\rho(\mathcal{A})$. This implies the resolvent estimate also for small λ. □

Now, we are ready to prove the main results on existence and uniqueness of strong T-periodic solutions to the bidomain equations with different kinds of ionic models. We remark at this point that the respective models treated in the following are slightly more general as described in Section 5.1. To be more precise, an additional parameter $\varepsilon > 0$ is introduced, that incorporates the phenomenon of fast and slow diffusion.

To this end, for each model we start by calculating the equilibrium points. After that, we write the solutions to the respective bidomain models as the sum of the equilibrium solution and a perturbation. Hence, we obtain an equation for the perturbation for which we apply Theorem 5.2.8 to show the existence and uniqueness of strong periodic solutions for suitable equilibrium points.

Additionally to Assumption 5.1.1 on the conductivity matrices of the bidomain operator A, due to Proposition 5.2.11 we require the following regularity and periodicity conditions on the forcing term I.

Assumption 5.2.13. *Let $2\theta \in (0,1)$, $1 \leq p < \infty$, and $1 < q < \infty$ satisfy $2\theta > d/q$ or, if $p = 1$ let $2\theta \geq d/q$. Assume $I : \mathbb{R} \to D_A(\theta, p)$ is a T-periodic function satisfying $I_{|(0,T)} \in \mathbb{F}_A$ for some $\theta \in (0, 1/2)$ and $T > 0$.*

We start with the most classical model due to FitzHugh and Nagumo.

The Bidomain FitzHugh–Nagumo Equation

For $T > 0$, $0 < a < 1$, and $b, c, \varepsilon > 0$, the periodic bidomain FitzHugh–Nagumo equations are given by

$$
(5.20) \qquad \begin{cases} \partial_t u + \varepsilon A u = I - \dfrac{1}{\varepsilon}[u^3 - (a+1)u^2 + au + w] & \text{in } \mathbb{R} \times \Omega, \\ \partial_t w = cu - bw & \text{in } \mathbb{R} \times \Omega, \\ u(t) = u(t+T) & \text{in } \mathbb{R} \times \Omega, \\ w(t) = w(t+T) & \text{in } \mathbb{R} \times \Omega. \end{cases}
$$

As described above, we start by calculating the equilibrium points. To do so, we consider

$$
(5.21) \qquad u^3 - (a+1)u^2 + au + w = 0,
$$

$$
(5.22) \qquad cu - bw = 0.
$$

Then, the equilibrium points are $(u_1, w_1) = (0, 0)$ and assuming $c < b\Big(\frac{(a+1)^2}{4} - a\Big)$, we obtain furthermore

$$(u_2, w_2) = \left(\frac{1}{2}(a+1-d), \frac{c}{2b}(a+1-d)\right), \tag{5.23}$$

$$(u_3, w_3) = \left(\frac{1}{2}(a+1+d), \frac{c}{2b}(a+1+d)\right), \tag{5.24}$$

with $d = \sqrt{(a+1)^2 - 4(a+\frac{c}{b})}$. In the following, we use the results from Subsections 5.2.1 and 5.2.2 to obtain periodic solutions in a neighborhood of some of these equilibrium points. In order to do so, we employ Taylor expansion at the equilibrium points and perform the following change of variables

$$\begin{pmatrix} v \\ z \end{pmatrix} := \begin{pmatrix} u - u_i \\ w - w_i \end{pmatrix}$$

for $i = 1, 2, 3$. Then, the nonlinearities describing the ionic transport read as follows

$$\begin{aligned} f(v, z) &= \frac{1}{\varepsilon}[v^3 + (3u_i - a - 1)v^2 + (3u_i^2 - 2(a+1)u_i + a)v + z], \\ g(v, z) &= -cv + bz. \end{aligned}$$

Plugging this into equation (5.20) and shifting the linear parts of f and g to the left-hand side yields

$$\begin{cases} \partial_t \begin{pmatrix} v \\ z \end{pmatrix} + \begin{pmatrix} \varepsilon A + \frac{1}{\varepsilon}[3u_i^2 - 2(a+1)u_i + a] & \frac{1}{\varepsilon} \\ -c & b \end{pmatrix} \begin{pmatrix} v \\ z \end{pmatrix} \\ \qquad\qquad = \begin{pmatrix} I - \frac{1}{\varepsilon}[v^3 + (3u_i - a - 1)v^2] \\ 0 \end{pmatrix}, \\ v(t) = v(t+T), \\ z(t) = z(t+T), \end{cases} \tag{5.25}$$

for $i = 1, 2, 3$. Due to Proposition 5.2.11 the nonlinearity in (5.25) satisfies Assumption 5.2.7.

It remains to check that the operator matrix in (5.25) is invertible and that the negative of this operator matrix generates a bounded analytic semigroup. This has to be done separately for the three equilibrium points. Starting with the equilibrium point $(0,0)$, the operator $-(\varepsilon A + \frac{a}{\varepsilon})$ generates a bounded analytic semigroup by Proposition 5.1.3 and since $0 \in \rho(\varepsilon A + \frac{a}{\varepsilon})$, we may apply Lemma 5.2.12 to conclude that the negative of the operator matrix in (5.25) has zero in its resolvent set and generates a bounded analytic semigroup. Thus, Theorem 5.2.8 is applicable in the case of the equilibrium point $(0,0)$ and delivers a unique strong periodic solution (v,z) to (5.25) in the desired function space for small periodic forcings I.

For the second equilibrium point we have $3u_2^2 - 2(a+1)u_2 + a < 0$. Since $0 \in \sigma(A)$ the operator $-(\varepsilon A + \frac{1}{\varepsilon}[3u_2^2 - 2(a+1)u_2 + a])$ does not generate a bounded analytic semigroup so that Lemma 5.2.12 is not applicable.

Considering the third equilibrium point and assuming that

$$u_3 > \frac{a + 1 + \sqrt{(a+1)^2 - 3a}}{3},$$

we obtain $3u_3^2 - 2(a+1)u_3 + a > 0$. Thus, $-(\varepsilon A + \frac{1}{\varepsilon}[3u_3^2 - 2(a+1)u_3 + a])$ generates a bounded analytic semigroup by Proposition 5.1.3 and $0 \in \rho(\varepsilon A + \frac{1}{\varepsilon}[3u_3^2 - 2(a+1)u_3 + a])$. Hence, we can apply Lemma 5.2.12 to conclude that the negative of the operator matrix in (5.25) has zero in its resolvent set and generates a bounded analytic semigroup. Thus, Theorem 5.2.8 is applicable in this case of the equilibrium point (u_3, w_3) and delivers a unique strong periodic solution (v,z) to (5.25) in the desired function spaces for small periodic forcings I.

Denoting the stability condition on the coefficients by

$$(\mathrm{S}_{\mathrm{FN}}) \quad c < b\left(\frac{(a-1)^2}{4} - a\right) \quad \text{and} \quad u_3 > \frac{1}{3}\left(a + 1 + \sqrt{(a+1)^2 - 3a}\right),$$

the considerations above prove the following theorem on existence and uniqueness of strong time-periodic solutions to the bidomain FitzHugh–Nagumo equations.

Theorem 5.2.14. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded C^2-domain and suppose that Assumptions 5.1.1 and 5.2.13 hold true.*

a) *Then there exist constants* $R > 0$ *and* $C(R) > 0$ *such that if* $\|I\|_{\mathbb{F}_A} < C(R)$*, the equation* (5.20) *admits a unique T-periodic strong solution* (u, w) *with* $(u, w)_{|(0,T)} \in \overline{B_{\mathbb{E}}}((0, 0), R)$.

b) *If condition* $(\mathrm{S_{FN}})$ *is satisfied, then there exist constants* $R > 0$ *and* $C(R) > 0$ *such that if* $\|I\|_{\mathbb{F}_A} < C(R)$*, the equation* (5.20) *admits a unique T-periodic strong solution* (u, w) *with* $(u, w)_{|(0,T)} \in \overline{B_{\mathbb{E}}}((u_3, w_3), R)$.

The Bidomain Aliev–Panfilov Equation

For $T > 0$, $0 < a < 1$, and $d, k, \varepsilon > 0$, the periodic bidomain Aliev–Panfilov equations are given by

(5.26)
$$\begin{cases} \partial_t u + \varepsilon A u = I - \dfrac{1}{\varepsilon}[ku^3 - k(a+1)u^2 + kau + uw] & \text{in } \mathbb{R} \times \Omega, \\ \partial_t w = -(ku(u-1-a) + dw) & \text{in } \mathbb{R} \times \Omega, \\ u(t) = u(t+T) & \text{in } \mathbb{R} \times \Omega, \\ w(t) = w(t+T) & \text{in } \mathbb{R} \times \Omega. \end{cases}$$

As before, we start by calculating the equilibrium points. To do so, we consider

$$ku^3 - k(a+1)u^2 + kau + uw = 0, \tag{5.27}$$
$$ku(u-1-a) + dw = 0. \tag{5.28}$$

Then, the equilibrium points are $(u_1, w_1) = (0, 0)$ and, if we assume $\frac{(a+1)^2}{4} + \frac{da}{1-d} > 0$, furthermore

$$(u_2, w_2) = \left(\frac{a+1}{2} - e, -ku_2^2 + k(a+1)u_2 - ka\right), \tag{5.29}$$
$$(u_3, w_3) = \left(\frac{a+1}{2} + e, -ku_3^2 + k(a+1)u_3 - ka\right). \tag{5.30}$$

with $e = \sqrt{\frac{(a+1)^2}{4} + \frac{da}{1-d}}$. In the following, we want to use the results from Subsections 5.2.1 and 5.2.2 to obtain periodic solutions in a neighborhood of some of these equilibrium points. In order to do so, we proceed as above and employ Taylor expansion at the equilibrium points and perform the following change of variables

$$\begin{pmatrix} v \\ z \end{pmatrix} := \begin{pmatrix} u - u_i \\ w - w_i \end{pmatrix}$$

for $i = 1, 2, 3$. Then, the nonlinearities describing the ionic transport read as follows

$$\begin{aligned} f(v, z) &= \frac{1}{\varepsilon}[kv^3 + (3ku_i - k(a+1))v^2 + (3ku_i^2 - 2k(a+1)u_i + ka + w_i)v \\ &\qquad + u_i z + vz], \\ g(v, z) &= (2ku_i - k(a+1))v + dz + kv^2. \end{aligned}$$

Plugging this into equation (5.26) and shifting the linear parts of f and g to the left-hand side yields

$$(5.31) \qquad \begin{cases} \partial_t \begin{pmatrix} v \\ z \end{pmatrix} + \begin{pmatrix} \varepsilon A + \frac{1}{\varepsilon}[3ku_i^2 - 2k(a+1)u_i + ka + w_i] & \frac{u_i}{\varepsilon} \\ 2ku_i - k(a+1) & d \end{pmatrix} \begin{pmatrix} v \\ z \end{pmatrix} \\ \qquad\qquad = \begin{pmatrix} I - \frac{1}{\varepsilon}[kv^3 + (3ku_i - k(a+1))v^2 + vz] \\ -kv^2 \end{pmatrix}, \\ v(t) = v(t+T), \\ z(t) = z(t+T), \end{cases}$$

for $i = 1, 2, 3$. Due to Proposition 5.2.11, the nonlinearity in (5.31) satisfies Assumption 5.2.7.

As for the FitzHugh–Nagumo system, we next have to check the operator matrix in (5.31) separately for the three equilibrium points. First, for the equilibrium point $(0, 0)$, the operator $-(\varepsilon A + \frac{ka}{\varepsilon})$ generates a bounded analytic semigroup by Proposition 5.1.3 and since $0 \in \rho(\varepsilon A + \frac{ka}{\varepsilon})$, we may apply Lemma 5.2.12 to conclude that the negative of the operator matrix in (5.31) has zero in its resolvent set and generates a bounded analytic semigroup. Thus, Theorem 5.2.8 is applicable in the case of the equilibrium point $(0, 0)$ and delivers a unique strong periodic solution (v, z) to (5.31) in the desired function space for small periodic forcings I.

For the second equilibrium point we see that $u_2 < 0$, so that the upper right component of the operator matrix is negative. Therefore, we cannot apply Lemma 5.2.12 for (u_2, w_2).

Similarly, for (u_3, w_3) it is

$$2ku_3 - k(a+1) = 2ke > 0.$$

Hence, Lemma 5.2.12 is not applicable in this case.

Summarizing the considerations above, we obtain the following theorem on existence and uniqueness of strong time-periodic solutions to the bidomain Aliev–Panfilov equations.

Theorem 5.2.15. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded C^2-domain and suppose that Assumptions 5.1.1 and 5.2.13 hold true. Then, there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}_A} < C(R)$, the equation* (5.26) *admits a unique T-periodic strong solution (u, w) with $(u, w)_{|(0,T)} \in \overline{B_{\mathbb{E}}}((0,0), R)$.*

The Bidomain Rogers–McCulloch Equation

For $T > 0$, $0 < a < 1$, and $b, c, d, \varepsilon > 0$, the periodic bidomain Rogers–McCulloch equations are given by

(5.32)
$$\begin{cases} \partial_t u + \varepsilon A u = I - \dfrac{1}{\varepsilon}[bu^3 - b(a+1)u^2 + bau + uw] & \text{in } \mathbb{R} \times \Omega, \\ \partial_t w = cu - dw & \text{in } \mathbb{R} \times \Omega, \\ u(t) = u(t+T) & \text{in } \mathbb{R} \times \Omega, \\ w(t) = w(t+T) & \text{in } \mathbb{R} \times \Omega. \end{cases}$$

As for the other models, we start by calculating the equilibrium points. To do so, we consider

$$bu^3 - b(a+1)u^2 + bau + uw = 0, \tag{5.33}$$
$$cu - dw = 0. \tag{5.34}$$

Then, the equilibrium points are $(u_1, w_1) = (0,0)$ and, if we assume $\left(a+1-\frac{c}{bd}\right)^2 - 4a > 0$, furthermore

$$(u_2, w_2) = \left(\frac{1}{2}\left(a+1-\frac{c}{bd}-e\right), \frac{c}{2d}\cdot\left(a+1-\frac{c}{bd}-e\right)\right), \tag{5.35}$$

$$(u_3, w_3) = \left(\frac{1}{2}\left(a+1-\frac{c}{bd}+e\right), \frac{c}{2d}\cdot\left(a+1-\frac{c}{bd}+e\right)\right). \tag{5.36}$$

with $e = \sqrt{\left(a+1-\frac{c}{bd}\right)^2 - 4a}$. In the following, we want to use the results from Subsections 5.2.1 and 5.2.2 to obtain periodic solutions in a neighborhood of some of these equilibrium points. In order to do so, we employ Taylor expansion at the equilibrium points and perform the following change of variables

$$\begin{pmatrix} v \\ z \end{pmatrix} := \begin{pmatrix} u - u_i \\ w - w_i \end{pmatrix}$$

for $i = 1, 2, 3$. Then, the nonlinearities describing the ionic transport read as follows

$$\begin{aligned} f(v,z) &= \frac{1}{\varepsilon}[bv^3 + (3bu_i - b(a+1))v^2 + (3bu_i^2 - 2b(a+1)u_i + ba + w_i)v \\ &\quad + u_i z + vz], \\ g(v,y) &= -cv + dz. \end{aligned}$$

Plugging this into equation (5.32) and shifting the linear parts of f and g to the left-hand side yields

$$\left\{\begin{aligned} &\partial_t \begin{pmatrix} v \\ z \end{pmatrix} + \begin{pmatrix} \varepsilon A + \frac{1}{\varepsilon}[3bu_i^2 - 2b(a+1)u_i + ba + w_i] & \frac{u_i}{\varepsilon} \\ -c & d \end{pmatrix} \begin{pmatrix} v \\ z \end{pmatrix} \\ &\qquad = \begin{pmatrix} I - \frac{1}{\varepsilon}[bv^3 + (3bu_i - b(a+1))v^2 + vz] \\ 0 \end{pmatrix}, \\ &v(t) = v(t+T), \\ &z(t) = z(t+T), \end{aligned}\right. \tag{5.37}$$

for $i = 1, 2, 3$. Due to Proposition 5.2.11, the nonlinearity in (5.37) satisfies Assumption 5.2.7.

As for the other models, we next have to check the operator matrix in (5.37) separately for the three equilibrium points. First, for the equilibrium point $(0, 0)$, the operator $-(\varepsilon A + \frac{ba}{\varepsilon})$ generates a bounded analytic semigroup by Proposition 5.1.3 and since $0 \in \rho(\varepsilon A + \frac{ba}{\varepsilon})$, we can apply Lemma 5.2.12 to conclude that the negative of the operator matrix in (5.37) has zero in its resolvent set and generates a bounded analytic semigroup. Thus, Theorem 5.2.8 is applicable in the case of the equilibrium point $(0, 0)$ and delivers a unique strong periodic solution (v, z) to (5.37) in the desired function space for small forcings I.

Next, equation (5.33) implies $w_i = -bu_i^2 + b(a + 1)u_i - ba$ for $i = 2, 3$. Then

$$3bu_i^2 - 2b(a + 1)u_i + ba + w_i = u_i(2bu_i - b(a + 1)).$$

Hence, for the second equilibrium point we either have $3bu_2^2 - 2b(a + 1)u_2 + ba + w_2 < 0$, then $-(\varepsilon A + \frac{1}{\varepsilon}[3bu_2^2 - 2b(a + 1)u_2 + ba + w_2])$ does not generate a bounded analytic semigroup, or $u_2 < 0$, then the upper right component of the operator matrix is negative. Therefore, we cannot apply Lemma 5.2.12 for (u_2, w_2).

Considering the third equilibrium point and assuming the following stability condition on the coefficients

$$(\mathrm{S_{RM}}) \qquad \sqrt{\left(a + 1 - \frac{c}{bd}\right)^2 - 4a} - \frac{c}{bd} > 0,$$

we obtain $3bu_3^2 - 2b(a + 1)u_3 + ba + w_3 > 0$ and $u_3 > 0$. Thus, $-(\varepsilon A + \frac{1}{\varepsilon}[3bu_3^2 - 2b(a+1)u_3 + ba + w_3])$ generates a bounded analytic semigroup by Proposition 5.1.3 and $0 \in \rho(\varepsilon A + \frac{1}{\varepsilon}[3bu_3^2 - 2b(a + 1)u_3 + ba + w_3])$. Hence, we can apply Lemma 5.2.12 to conclude that the negative of the operator matrix in (5.37) has zero in its resolvent and generates a bounded analytic semigroup. Thus, Theorem 5.2.8 is applicable in this case for (u_3, w_3) and delivers a unique strong periodic solution (v, z) in the desired function space for small forcings I.

Summarizing, the considerations above prove the following theorem on existence and uniqueness of strong time-periodic solutions to the bidomain Rogers–McCulloch equations.

Theorem 5.2.16. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded C^2-domain and suppose that Assumptions 5.1.1 and 5.2.13 hold true.*

a) *Then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}_A} < C(R)$, the equation* (5.32) *admits a unique T-periodic strong solution (u, w) with $(u, w)_{|(0,T)} \in \overline{B_{\mathbb{E}}}((0, 0), R)$.*

b) *If condition* ($\mathrm{S_{RM}}$) *is satisfied, then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}_A} < C(R)$, the equation* (5.32) *admits a unique T-periodic strong solution (u, w) with $(u, w)_{|(0,T)} \in \overline{B\mathbb{E}}((u_3, w_3), R)$.*

The Bidomain Allen–Cahn Equation

For $T > 0$, the periodic bidomain Allen–Cahn equation is given by

$$\begin{cases} \partial_t u + Au = I + u - u^3 & \text{in } \mathbb{R} \times \Omega, \\ u(t) = u(t + T) & \text{in } \mathbb{R} \times \Omega. \end{cases} \tag{5.38}$$

The equilibrium points of this system are $u_1 = -1$, $u_2 = 0$, and $u_3 = 1$. In the following, we want to use the results from Subsections 5.2.1 and 5.2.2 to obtain periodic solutions in a neighborhood of some of these equilibrium points. In order to do so, we employ Taylor expansion at the equilibrium points and perform the change of variables $v = u - u_i$ for $i = 1, 2, 3$. Then, the function $f(u) = u^3 - u$ reads as follows

$$f(v) = v^3 + 3u_i v^2 - (1 - 3u_i^2)v.$$

Plugging this into equation (5.38) and shifting the linear parts of f to the left-hand side yields

$$\begin{cases} \partial_t v + (A - 1 + 3u_i^2)v = I - v^3 - 3u_i v^2 & \text{in } \mathbb{R} \times \Omega, \\ u(t) = u(t + T) & \text{in } \mathbb{R} \times \Omega \end{cases} \tag{5.39}$$

for $i = 1, 2, 3$. Due to Proposition 5.2.11, the nonlinearity in (5.39) satisfies Assumption 5.2.7. Since $-(A+2)$ generates a bounded analytic semigroup by Proposition 5.1.3 and since $0 \in \rho(A + 2)$, Theorem 5.2.8 is applicable in the case of the equilibrium points u_1 and u_3 and delivers a unique strong periodic solution v to (5.39) in the desired function space for small forcings I. Summarizing the considerations above, we obtain the following theorem on existence and uniqueness of strong time-periodic solutions to the bidomain Allen-Cahn equation.

Theorem 5.2.17. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded C^2-domain and suppose that Assumptions 5.1.1 and 5.2.13 hold true.*

a) Then, there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}_A} < C(R)$ the equation (5.38) admits a unique T-periodic strong solution u with $u_{|(0,T)} \in \overline{B_{\mathbb{E}_A^{\mathrm{per}}}}(-1, R)$.

b) Then, there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}_A} < C(R)$ the equation (5.38) admits a unique T-periodic strong solution u with $u_{|(0,T)} \in \overline{B_{\mathbb{E}_A^{\mathrm{per}}}}(1, R)$.

5.2.4 An Alternative Approach by the Semilinear Arendt–Bu Theorem

In this subsection, we show the existence of a unique strong time-periodic solution to the bidomain equations by a different approach than in the previous subsections. Instead of the periodic version of the Da Prato–Grisvard theorem (Theorem 5.2.3), we apply the semilinear version of the Arendt–Bu theorem (Proposition 2.5.5) to solve the linearized system. This means that we use the property of maximal regularity in place of the weaker property, that the involved operator is the generator of a bounded analytic semigroup. Hence, we obtain results in the usual maximal regularity spaces, where the underlying Banach space is $L^q(\Omega)$ instead of the real interpolation space $D_A(\theta, p)$. However, by this approach we are not able to handle the endpoint case $p = 1$.

To be more precise, in the following we consider the spaces

$$\begin{aligned} X_0 &:= L^q(\Omega) \times L^q(\Omega), \\ \mathbb{E}_1 &:= L^p(0,T; W^{2,q}_N(\Omega)) \cap W^{1,p}(0,T; L^q(\Omega)), \\ \mathbb{E}_2 &:= W^{1,p}(0,T; L^q(\Omega)), \end{aligned} \tag{5.40}$$

as well as

$$\mathbb{F} := L^p(0,T; X_0) \text{ and } \mathbb{E} := \mathbb{E}_1 \times \mathbb{E}_2. \tag{5.41}$$

The Bidomain FitzHugh–Nagumo Equation

Recall, that for $T > 0$, $0 < a < 1$, and $b, c, \varepsilon > 0$, the periodic bidomain FitzHugh–Nagumo equations are given by

$$
\left\{
\begin{aligned}
\partial_t u + \varepsilon A u &= I - \frac{1}{\varepsilon}[u^3 - (a+1)u^2 + au + w] && \text{in } \mathbb{R} \times \Omega, \\
\partial_t w &= cu - bw && \text{in } \mathbb{R} \times \Omega, \\
u(t) &= u(t+T) && \text{in } \mathbb{R} \times \Omega, \\
w(t) &= w(t+T) && \text{in } \mathbb{R} \times \Omega.
\end{aligned}
\right.
\tag{5.42}
$$

As in Subsection 5.2.3, we rewrite this system by calculating the equilibrium points. Using the same notation as before, we obtain the system

$$
\left\{
\begin{aligned}
\partial_t \begin{pmatrix} v \\ z \end{pmatrix} + \begin{pmatrix} \varepsilon A + \frac{1}{\varepsilon}[3u_i^2 - 2(a+1)u_i + a] & \frac{1}{\varepsilon} \\ -c & b \end{pmatrix} \begin{pmatrix} v \\ z \end{pmatrix} & \\
= \begin{pmatrix} I - \frac{1}{\varepsilon}[v^3 + (3u_i - a - 1)v^2] \\ 0 \end{pmatrix}&, \\
v(t) = v(t+T)&, \\
z(t) = z(t+T)&,
\end{aligned}
\right.
\tag{5.43}
$$

for $i = 1, 2, 3$, where u_i and w_i are given by (5.23), (5.24), and $(u_1, w_1) = (0, 0)$. In order to apply the theory introduced in Subsection 2.5.3 to the system above, we define on X_0 the operators $\mathcal{A}_i$ and for $y = (v, z)$ the mapping F as

$$
\begin{aligned}
-\mathcal{A}_i &:= \begin{pmatrix} \varepsilon A + \frac{1}{\varepsilon}[3u_i^2 - 2(a+1)u_i + a] & \frac{1}{\varepsilon} \\ -c & b \end{pmatrix} \\
F_i(t, y) &:= \begin{pmatrix} I - \frac{1}{\varepsilon}[v^3 + (3u_i - a - 1)v^2] \\ 0 \end{pmatrix}.
\end{aligned}
$$

Then, the periodic bidomain FitzHugh–Nagumo equations correspond to the equation

$$
\left\{
\begin{aligned}
\partial_t y(t) - \mathcal{A}_i y(t) &= F_i(t, y(t)) \quad t \in (0, T), \\
y(0) &= y(T).
\end{aligned}
\right.
$$

Provided that

$$p, q \in (1, \infty) \quad \text{such that} \quad \frac{2}{3p} + \frac{d}{3q} < 1 \tag{5.44}$$

the result on existence and uniqueness of strong time-periodic solutions to the bidomain FitzHugh–Nagumo equations reads as follows.

Theorem 5.2.18. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded C^2-domain, $T > 0$, and suppose that* (5.44) *and Assumption 5.1.1 hold true. Let $I \in \mathbb{F}$ be T-periodic.*

a) Then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}} < C(R)$, the equation (5.20) *admits a unique T-periodic strong solution (u, w) with $(u, w)_{|(0,T)} \in \overline{B_{\mathbb{E}}}((0, 0), R)$.*

b) If condition ($\mathrm{S_{FN}}$) *is satisfied, then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}} < C(R)$, the equation* (5.20) *admits a unique T-periodic strong solution (u, w) with $(u, w)_{|(0,T)} \in \overline{B_{\mathbb{E}}}((u_3, w_3), R)$.*

Proof. In order to prove the theorem, by Proposition 2.5.5 it suffices to show the following properties:

a) The operators $\mathcal{A}_1$ and $\mathcal{A}_3$ admit maximal periodic L^p-regularity on X_0.

b) $F_1(\cdot, y(\cdot)) \in \mathbb{F}$ and $F_3(\cdot, (y(\cdot)) \in \mathbb{F}$ for any $y \in \mathbb{E}$.

c) There exists a $C > 0$ such that for any $R > 0$ it holds

$$\|F_i(\cdot, y(\cdot)) - F_i(\cdot, \tilde{y}(\cdot))\|_{\mathbb{F}} \leq CR\|y - \tilde{y}\|_{\mathbb{E}}$$

for all $y, \tilde{y} \in \overline{B_{\mathbb{E}}}((u_i, w_i), R)$ for $i = 1, 3$.

As described in [42, Section 2], the operator $\mathcal{A}_i$ admits maximal L^p-regularity, provided $\frac{1}{\varepsilon}[3u_i^2 - 2(a + 1)u_i + a] > 0$. In the same way as in the proof of Theorem 5.2.14 we show that this is true for the equilibrium points (u_1, w_1) and (u_3, w_3). Since in these cases the operators are also invertible, Proposition 2.2.1 yields that $\mathcal{A}_1$ and $\mathcal{A}_3$ admit maximal periodic L^p-regularity on X_0, which proves a).

Next, we show that $F_i(\cdot, y(\cdot)) \in \mathbb{F}$ is satisfied for $y \in \mathbb{E}$. Since u_i is constant, Hölder's inequality yields

$$\begin{aligned}
\|F_i(\cdot, y(\cdot))\|_{\mathbb{F}} &\leq c\left(\|v^3\|_{L^p(0,T;L^q(\Omega))} + \|v^2\|_{L^p(0,T;L^q(\Omega))}\right) + \|I\|_{L^p(0,T;L^q(\Omega))} \\
&\leq c\left(\|v\|^3_{L^{3p}(0,T;L^{3q}(\Omega))} + \|v\|^2_{L^{2p}(0,T;L^{2q}(\Omega))}\right) + \|I\|_{L^p(0,T;L^q(\Omega))}.
\end{aligned}$$

Next, due to Proposition 2.5.1 we have

$$\mathbb{E}_1 \hookrightarrow H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega)) \tag{5.45}$$

for any $\theta \in (0,1)$. Furthermore, with (5.44) by Sobolev embeddings we obtain

$$H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega)) \hookrightarrow L^{3p}(0,T;L^{3q}(\Omega)). \tag{5.46}$$

Applying these embeddings to the inequality above, we obtain

$$\begin{aligned}
\|F_i(\cdot, y(\cdot))\|_{\mathbb{F}} &\leq c\left(\|v\|^3_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} + \|v\|^2_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\right) + \|I\|_{\mathbb{F}} \\
&\leq c\left(\|v\|^3_{\mathbb{E}_1} + \|v\|^2_{\mathbb{E}_1}\right) + \|I\|_{\mathbb{F}} \\
&\leq c\left(\|y\|^3_{\mathbb{E}} + \|y\|^2_{\mathbb{E}}\right) + \|I\|_{\mathbb{F}}.
\end{aligned}$$

This proves b).

It remains to show assertion c). Therefore, firstly let $y, \tilde{y} \in \overline{B_{\mathbb{E}}}((0,0),R)$. Then, Hölder's inequality implies

$$\begin{aligned}
&\|F_1(\cdot, y(\cdot)) - F_1(\cdot, \tilde{y}(\cdot))\|_{\mathbb{F}} \\
&\quad\leq c\left(\|v^3 - \tilde{v}^3\|_{L^p(0,T;L^q(\Omega))} + \|v^2 - \tilde{v}^2\|_{L^p(0,T;L^q(\Omega))}\right) \\
&\quad\leq c\left(\|(v^2 + v\tilde{v} + \tilde{v}^2)(v - \tilde{v})\|_{L^p(0,T;L^q(\Omega))} + \|(v + \tilde{v})(v - \tilde{v})\|_{L^p(0,T;L^q(\Omega))}\right) \\
&\quad\leq c\Big(\|v^2 + v\tilde{v} + \tilde{v}^2\|_{L^{3p/2}(0,T;L^{3q/2}(\Omega))}\|v - \tilde{v}\|_{L^{3p}(0,T;L^{3q}(\Omega))} \\
&\qquad+\|v + \tilde{v}\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|v - \tilde{v}\|_{L^{2p}(0,T;L^{2q}(\Omega))}\Big) \\
&\quad\leq c\Big((\|v\|^2_{L^{3p}(0,T;L^{3q}(\Omega))} + \|v\|_{L^{3p}(0,T;L^{3q}(\Omega))}\|\tilde{v}\|_{L^{3p}(0,T;L^{3q}(\Omega))} \\
&\qquad+ \|\tilde{v}\|^2_{L^{3p}(0,T;L^{3q}(\Omega))})\|(v - \tilde{v})\|_{L^{3p}(0,T;L^{3q}(\Omega))} \\
&\qquad+\|v + \tilde{v}\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|v - \tilde{v}\|_{L^{2p}(0,T;L^{2q}(\Omega))}\Big).
\end{aligned}$$

Next, we apply the Sobolev embedding (5.46) and the mixed derivative theorem (5.45) to obtain

$$\begin{aligned}
&\|F_1(\cdot, y(\cdot)) - F_1(\cdot, \tilde{y}(\cdot))\|_{\mathbb{F}} \\
&\quad \le c\Big(\|v\|^2_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} + \|v\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|\tilde{v}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \\
&\qquad + \|\tilde{v}\|^2_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\Big)\|v - \tilde{v}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \\
&\qquad + \|v + \tilde{v}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|v - \tilde{v}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\Big) \\
&\quad \le c\Big(\|v\|^2_{\mathbb{E}_1} + \|v\|_{\mathbb{E}_1}\|\tilde{v}\|_{\mathbb{E}_1} + \|\tilde{v}\|^2_{\mathbb{E}_1})\|(v-\tilde{v})\|_{\mathbb{E}_1} + \|v+\tilde{v}\|_{\mathbb{E}_1}\|v-\tilde{v}\|_{\mathbb{E}_1}\Big) \\
&\quad \le c(R^2 + R)\|y - \tilde{y}\|_{\mathbb{E}}.
\end{aligned}$$

This proves c) for the first equilibrium point. For (u_3, w_3) the Lipschitz estimate can be done in the same way. Hence, the proof is finished. □

The Bidomain Aliev–Panfilov Equation

Recall, that for $T > 0$, $0 < a < 1$, and $d, k, \varepsilon > 0$, the periodic bidomain Aliev–Panfilov equations are given by

(5.47)
$$\begin{cases}
\partial_t u + \varepsilon A u = I - \dfrac{1}{\varepsilon}[ku^3 - k(a+1)u^2 + kau + uw] & \text{in } \mathbb{R} \times \Omega, \\
\partial_t w = -(ku(u - 1 - a) + dw) & \text{in } \mathbb{R} \times \Omega, \\
u(t) = u(t+T) & \text{in } \mathbb{R} \times \Omega, \\
w(t) = w(t+T) & \text{in } \mathbb{R} \times \Omega.
\end{cases}$$

As described in Subsection 5.2.3, for the equilibrium points (u_2, w_2) and (u_3, w_3) we cannot expect to obtain periodic solutions by our approach. Hence, we will focus on the equilibrium point $(0, 0)$ in the following. In order to apply the theory introduced in Subsection 2.5.3 to the system above, we define on X_0 the operators $\mathcal{A}$ and for $y = (u, w)$ the mapping F as

$$-\mathcal{A} := \begin{pmatrix} \varepsilon A + \frac{ka}{\varepsilon} & 0 \\ -k(a+1) & d \end{pmatrix}$$

$$F(t, y) := \begin{pmatrix} I - \frac{1}{\varepsilon}[u^3 - k(a+1)u^2 + uw] \\ -ku^2 \end{pmatrix}.$$

Then, the periodic bidomain Aliev–Panfilov equations correspond to the equation

$$\begin{cases} \partial_t y(t) - \mathcal{A}y(t) = F(t, y(t)) & t \in (0,T), \\ \qquad\qquad y(0) = y(T). \end{cases}$$

Provided that

$$p \in (1,\infty),\ q \in (d,\infty) \quad \text{such that} \quad \frac{2}{3p} + \frac{d}{3q} < 1 \tag{5.48}$$

the result on existence and uniqueness of strong time-periodic solutions to the bidomain Aliev–Panfilov equations reads as follows.

Theorem 5.2.19. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded C^2-domain, $T > 0$, and suppose that* (5.48) *and Assumption 5.1.1 hold true. Let $I \in \mathbb{F}$ be T-periodic. Then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}} < C(R)$, the equation* (5.26) *admits a unique T-periodic strong solution (u,w) with $(u,w)_{|(0,T)} \in \overline{B_{\mathbb{E}}}((0,0),R)$.*

Proof. In order to prove the theorem, by Proposition 2.5.5 it suffices to show the following properties:

a) The operator $\mathcal{A}$ admits maximal periodic L^p-regularity on X_0.

b) $F(\cdot, y(\cdot)) \in \mathbb{F}$ for any $y \in \mathbb{E}$.

c) There exists a $C > 0$ such that for any $R > 0$ it holds

$$\|F(\cdot, y(\cdot)) - F(\cdot, \tilde{y}(\cdot))\|_{\mathbb{F}} \leq CR\|y - \tilde{y}\|_{\mathbb{E}}$$

for all $y, \tilde{y} \in \overline{B_{\mathbb{E}}}((0,0),R)$.

As described in [42, Section 2], the operator $\mathcal{A}$ admits the property of maximal L^p-regularity, provided $ka/\varepsilon > 0$, which is true due to the assumptions on the constants. Since the operator is also invertible, Proposition 2.2.1 yields that $\mathcal{A}$ admits maximal periodic L^p-regularity on X_0, which proves a).

Next, we show that $F(\cdot, y(\cdot)) \in \mathbb{F}$ is satisfied for $y \in \mathbb{E}$. By Hölder's inequality, we obtain

$$\begin{aligned}
\|F(\cdot, y(\cdot))\|_{\mathbb{F}} \leq c &\left(\|u^3\|_{L^p(0,T;L^q(\Omega))} + \|u^2\|_{L^p(0,T;L^q(\Omega))} + \|uw\|_{L^p(0,T;L^q(\Omega))}\right) \\
&+ \|I\|_{L^p(0,T;L^q(\Omega))} \\
\leq c&\Big(\|u\|^3_{L^{3p}(0,T;L^{3q}(\Omega))} + \|u\|^2_{L^{2p}(0,T;L^{2q}(\Omega))} \\
&+ \|u\|_{L^p(0,T;L^\infty(\Omega))}\|w\|_{L^\infty(0,T;L^q(\Omega))}\Big) + \|I\|_{L^p(0,T;L^q(\Omega))}
\end{aligned}$$

Due to (5.48) and Proposition 2.5.1 the embeddings (5.46) and (5.45) are valid. Furthermore, since $p > 1$ and $q > d$ in addition we have the continuous embeddings

$$W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{and} \quad W^{1,p}(0,T;L^q(\Omega)) \hookrightarrow L^\infty(0,T;L^q(\Omega)). \tag{5.49}$$

Applying these embeddings to the inequality above, we obtain

$$\begin{aligned}
\|F(\cdot, y(\cdot))\|_{\mathbb{F}} \leq c&\Big(\|u\|^3_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} + \|u\|^2_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \\
&+ \|u\|_{L^p(0,T;W^{1,q}(\Omega))}\|w\|_{W^{1,p}(0,T;L^q(\Omega))}\Big) + \|I\|_{\mathbb{F}} \\
\leq c &\left(\|u\|^3_{\mathbb{E}_1} + \|u\|^2_{\mathbb{E}_1} + \|u\|_{\mathbb{E}_1}\|w\|_{\mathbb{E}_2}\right) + \|I\|_{\mathbb{F}} \\
\leq c &\left\|y\|^3_{\mathbb{E}} + \|y\|^2_{\mathbb{E}}\right) + \|I\|_{\mathbb{F}}.
\end{aligned}$$

This proves b).

It remains to show assertion c). Therefore, let $y, \tilde{y} \in \overline{B_{\mathbb{E}}}((0,0), R)$. Then, Hölder's inequality implies

$$\begin{aligned}
&\|F(\cdot, y(\cdot)) - F(\cdot, \tilde{y}(\cdot))\|_{\mathbb{F}} \\
&\quad\leq c\Big(\|u^3 - \tilde{u}^3\|_{L^p(0,T;L^q(\Omega))} + \|u^2 - \tilde{u}^2\|_{L^p(0,T;L^q(\Omega))} \\
&\qquad + \|uw - \tilde{u}\tilde{w}\|_{L^p(0,T;L^q(\Omega))}\Big) \\
&\quad\leq c\Big(\|(u^2 + u\tilde{u} + \tilde{u}^2)(u - \tilde{u})\|_{L^p(0,T;L^q(\Omega))} + \|(u + \tilde{u})(u - \tilde{u})\|_{L^p(0,T;L^q(\Omega))} \\
&\qquad + \|uw - \tilde{u}w + \tilde{u}w - \tilde{u}\tilde{w}\|_{L^p(0,T;L^q(\Omega))}\Big) \\
&\quad\leq c\left(\|u^2 + u\tilde{u} + \tilde{u}^2\|_{L^{3p/2}(0,T;L^{3q/2}(\Omega))}\|u - \tilde{u}\|_{L^{3p}(0,T;L^{3q}(\Omega))}\right. \\
&\qquad + \|u + \tilde{u}\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|u - \tilde{u}\|_{L^{2p}(0,T;L^{2q}(\Omega))} \\
&\qquad \left. + \|(u - \tilde{u})w + \tilde{u}(w - \tilde{w})\|_{L^p(0,T;L^q(\Omega))}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq c\Big(\big(\|u\|^2_{L^{3p}(0,T;L^{3q}(\Omega))} + \|u\|_{L^{3p}(0,T;L^{3q}(\Omega))}\|\tilde{u}\|_{L^{3p}(0,T;L^{3q}(\Omega))} \\
&\quad + \|\tilde{u}\|^2_{L^{3p}(0,T;L^{3q}(\Omega))}\big)\|(u-\tilde{u})\|_{L^{3p}(0,T;L^{3q}(\Omega))} \\
&\quad + \|u+\tilde{u}\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|u-\tilde{u}\|_{L^{2p}(0,T;L^{2q}(\Omega))} \\
&\quad + \|u-\tilde{u}\|_{L^{p}(0,T;L^{\infty}(\Omega))}\|w\|_{L^{\infty}(0,T;L^{q}(\Omega))} \\
&\quad + \|\tilde{u}\|_{L^{p}(0,T;L^{\infty}(\Omega))}\|w-\tilde{w}\|_{L^{\infty}(0,T;L^{q}(\Omega))}\Big)
\end{aligned}
$$

Next, we apply the Sobolev embeddings (5.46) and (5.49) as well as the mixed derivative theorem (5.45) to obtain

$$
\begin{aligned}
&\|F(\cdot,y(\cdot)) - F(\cdot,\tilde{y}(\cdot))\|_{\mathbb{F}} \\
&\leq c\Big(\big(\|u\|^2_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} + \|u\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|\tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \\
&\quad + \|\tilde{u}\|^2_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\big)\|u-\tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \\
&\quad + \|u+\tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}\|u-\tilde{u}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \\
&\quad + \|u-\tilde{u}\|_{L^{p}(0,T;W^{1,q}(\Omega))}\|w\|_{W^{1,p}(0,T;L^{q}(\Omega))} \\
&\quad + \|\tilde{u}\|_{L^{p}(0,T;W^{1,q}(\Omega))}\|w-\tilde{w}\|_{W^{1,p}(0,T;L^{q}(\Omega))}\Big) \\
&\leq c\Big(\big(\|u\|^2_{\mathbb{E}_1} + \|u\|_{\mathbb{E}_1}\|\tilde{u}\|_{\mathbb{E}_1} + \|\tilde{u}\|^2_{\mathbb{E}_1}\big)\|(u-\tilde{u})\|_{\mathbb{E}_1} + \|u+\tilde{u}\|_{\mathbb{E}_1}\|u-\tilde{u}\|_{\mathbb{E}_1} \\
&\quad + \|u-\tilde{u}\|_{\mathbb{E}_1}\|w\|_{\mathbb{E}_2} + \|\tilde{u}\|_{\mathbb{E}_1}\|w-\tilde{w}\|_{\mathbb{E}_2}\Big) \\
&\leq c(R^2+R)\|y-\tilde{y}\|_{\mathbb{E}}.
\end{aligned}
$$

This proves c) and therefore the proof is finished. □

The Bidomain Rogers–McCulloch Equation

For the periodic bidomain Rogers–McCulloch equations (5.32) we proceed in a similar way as described above for the FitzHugh–Nagumo or Aliev-Panfilov equations to apply the theory introduced in Subsection 2.5.3. Using the reasoning as in the proofs of Theorem 5.2.18 and 5.2.19 combined with the arguments given in Subsection 5.2.3 concerning the Rogers–McCulloch model, we obtain the following theorem.

Theorem 5.2.20. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded C^2-domain, $T > 0$, and suppose that* (5.48) *and Assumption 5.1.1 hold true. Let $I \in \mathbb{F}$ be T-periodic.*

a) Then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}} < C(R)$, the equation (5.32) *admits a unique T-periodic strong solution (u, w) with $(u, w)_{|(0,T)} \in \overline{B_{\mathbb{E}}}((0, 0), R)$.*

b) If condition (S_{RM}) *is satisfied, then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}} < C(R)$, the equation* (5.32) *admits a unique T-periodic strong solution (u, w) with $(u, w)_{|(0,T)} \in \overline{B_{\mathbb{E}}}((u_3, w_3), R)$, where (u_3, w_3) is given as in* (5.36).

The Bidomain Allen–Cahn Equation

For the periodic bidomain Allen–Cahn equation (5.38) we combine the reasoning given in the proof of Theorem 5.2.18 with the arguments from Subsection 5.2.3 to obtain the following theorem.

Theorem 5.2.21. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded C^2-domain, $T > 0$, and suppose that* (5.44) *and Assumption 5.1.1 hold true. Let $I \in \mathbb{F}$ be T-periodic.*

a) Then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}} < C(R)$, the equation (5.38) *admits a unique T-periodic strong solution u with $u_{|(0,T)} \in \overline{B_{\mathbb{E}_1}}(-1, R)$.*

b) Then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}} < C(R)$, the equation (5.38) *admits a unique T-periodic strong solution u with $u_{|(0,T)} \in \overline{B_{\mathbb{E}_1}}(1, R)$.*

5.3 Strong Time-Periodic Solutions with Arbitrary Large Forces

The aim of this section is to prove the existence of time-periodic solutions to the bidomain equations without assuming any smallness condition on the external forces. We employ the method given by Galdi, Hieber, and Kashiwabara [29] for the case of the primitive equations.

For this purpose, we first show the existence of weak time-periodic solutions for a large class of nonlinear dynamic models, including those of FitzHugh–Nagumo, Aliev–Panfilov, and Rogers–McCulloch. This is done by combining a Galerkin approximation with Brouwer's fixed point theorem. Note that we will look at the Aliev–Panfilov model in a slightly modified form as considered, e.g., in [31]. Then, we use the global well-posedness result by Colli Franzone and Savaré [19] and consider the *weak* time-periodic solution as a *weak* solution to the initial value problem. Finally, we apply a weak-strong uniqueness argument to get a strong-time periodic solution without assuming any smallness condition for the externally applied currents in case of the FitzHugh–Nagumo model.

5.3.1 Weak Time-Periodic Solutions

In this subsection, we show the existence of weak time-periodic solutions by using a Galerkin approximation combined with Brouwer's fixed point theorem.

Let $T > 0$. We consider the abstract periodic formulation of the bidomain equations

$$\text{(PABDE)} \qquad \begin{cases} u' + Au + f(u,w) = I & \text{in } \mathbb{R} \times \Omega, \\ w' + g(u,w) = 0 & \text{in } \mathbb{R} \times \Omega, \\ u(t+T,x) = u(t,x) & \text{in } \mathbb{R} \times \Omega, \\ w(t+T,x) = w(t,x) & \text{in } \mathbb{R} \times \Omega, \end{cases}$$

where I is a T-periodic function.

Recall the abbreviations

$$V = H^1(\Omega), \; H = L^2(\Omega), \; V' = (H^1(\Omega))',$$

and $Q = \Omega \times (0,T)$. For the nonlinear functions f and g we assume the following growth conditions.

Assumption 5.3.1. *Let $p > 1$ be a number so that the Sobolev embedding $V \hookrightarrow L^p(\Omega)$ holds. In other words, $2 \leq p$ if $d = 2$; or $2 \leq p \leq 6$ if $d = 3$. The nonlinear terms $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are of the form*

$$\begin{aligned} f(u,w) &= f_1(u) + f_2(u)w, \\ g(u,w) &= g_1(u) + g_2 w, \end{aligned}$$

where $g_2 \in \mathbb{R}$ and $f_1, f_2, g_1 : \mathbb{R} \to \mathbb{R}$ are continuous functions. The functions are assumed to satisfy that there exist constants $C_0 \in \mathbb{R}$, $C_i > 0$ $(i = 1, \dots, 5)$ and $r > 0$ such that

$$C_0 + C_1|u|^p + C_2|w|^2 \le rf(u,w)u + g(u,w)w \tag{5.50}$$

$$|f_1(u)| \le C_3(1 + |u|^{p-1}) \tag{5.51}$$

$$|f_2(u)| \le C_4(1 + |u|^{p/2-1}) \tag{5.52}$$

$$|g_1(u)| \le C_5(1 + |u|^{p/2}) \tag{5.53}$$

for all $u, w \in \mathbb{R}$.

This assumption is a modified version of the assumption used in [13]. The nonlinearities by FitzHugh–Nagumo, Rogers–McCulloch, and Aliev–Panfilov in the modified form are covered by this assumption. We check this at the end of this section. Next, due to [13, Lemma 25] we obtain for any $(u, w) \in L^p(\Omega) \times H$ the following inequalities

$$\|f(u,w)\|_{p'}^{p'} \le C_6(1 + \|u\|_p^p + \|w\|_H^2)$$
$$\|g(u,w)\|_H^2 \le C_7(1 + \|u\|_p^p + \|w\|_H^2)$$

for some $C_i > 0$ $(i = 6, 7)$ depending on p and $C_3, \dots, C_5$, where p' is the Hölder conjugate exponent, i.e., $1/p + 1/p' = 1$. In particular, $f(u,w) \in L^{p'}(Q)$ and $g(u,w) \in L^2(Q)$ for all $u \in L^p(Q), w \in L^2(Q)$.

Under this assumption, weak time-periodic solutions for (PABDE) are defined as follows.

Definition 5.3.2. Let $T > 0$ and $I \in L^2(0,T;V')$ be T-periodic. Suppose that the Assumption 5.3.1 holds. Then a pair of (u, w) of $u : \mathbb{R} \times \Omega \to \mathbb{R}$, $w : \mathbb{R} \times \Omega \to \mathbb{R}$ is called a weak T-periodic solution to (ABDE) if

(i) $u \in C_w(0,T;H) \cap L^2(0,T;V) \cap L^p(Q)$, $w \in C_w(0,T;H)$,

(ii) For all $\varphi_1 \in W^{1,2}(0,T;H) \cap L^2(0,T;V) \cap L^p(Q)$ and all $\varphi_2 \in W^{1,2}(0,T;H)$,

$$\begin{aligned}&\int_0^t \{(u, \partial_t\varphi_1) - a(u,\varphi_1) - {}_{p'}\langle f(u,w), \varphi_1\rangle_p\}\,\mathrm{d}\tau \\ &\qquad = -\int_0^t {}_{V'}\langle I, \varphi_1\rangle_V\,\mathrm{d}\tau + (u(t),\varphi_1(t)) - (u(0),\varphi_1(0)),\\ &\int_0^t \{(w, \partial_t\varphi_2) - (g(u,w),\varphi_2)\}\,\mathrm{d}\tau = (w(t),\varphi_2(t)) - (w(0),\varphi_2(0)),\end{aligned}$$

for all $t \in (0,T)$.

(iii) $u(t+T,x) = u(t,x)$ and $w(t+T,x) = w(t,x)$ for all $t \in \mathbb{R}$ and almost all $x \in \Omega$.

Here $(\cdot,\cdot)$ denotes the L^2-inner product and $C_w(0,T;H)$ denotes the space of all weakly continuous functions u on $(0,T)$ with values in H, i.e., $u : (0,T) \to H$ such that $(u(t),\psi)$ is continuous in t for all $\psi \in H$.

A weak T-periodic solution (u,w) is called strong if, in addition to above, it holds

$$u \in W^{1,2}(0,T;H) \cap L^2(0,T;H^2(\Omega)), w \in W^{1,2}(0,T;H).$$

Then, we obtain the following result on existence of weak time-periodic solutions to (PABDE).

Theorem 5.3.3. *Let $T > 0$, $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded C^2-domain, and suppose that Assumptions 5.1.1 and 5.3.1 hold true. Then, for every T-periodic function $I \in L^2(0,T;V')$ there exists at least one weak T-periodic solution (u,w) to* (PABDE).

Proof. Let $\{\psi_i\}_{i=0}^\infty \subset V$ be the orthonormal basis of eigenvectors of the bidomain bilinear form a in H and let $\{\lambda_i\}_{i=0}^\infty \subset \mathbb{R}_{\geq 0}$ be the corresponding eigenvalues as in Lemma 5.1.2. Define

$$u_k(t,x) := \sum_{i=0}^{k} \alpha_{ki}(t)\psi_i(x), \tag{5.54}$$

$$w_k(t,x) := \sum_{i=0}^{k} \beta_{ki}(t)\psi_i(x), \tag{5.55}$$

where $\alpha_k(t) = \{\alpha_{kj}(t)\}_{j=0}^k$, $\beta_k(t) = \{\beta_{kj}(t)\}_{j=0}^k$ are the solutions of the system of the ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\alpha_{kj} = -\alpha_{kj}\lambda_j - \displaystyle\int_\Omega f(u_k,w_k)\psi_j \,\mathrm{d}x + {}_{V'}\langle I(t),\psi_j\rangle_V, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\beta_{kj} = -\displaystyle\int_\Omega g(u_k,w_k)\psi_j \,\mathrm{d}x, \\ \alpha_{kj}(0) = a_j, \\ \beta_{kj}(0) = b_j, \end{cases} \tag{5.56}$$

for $j = 0, 1, \ldots, k$. The initial values $\mathbf{a}_k = \{a_j\}_{j=0}^k \in \mathbb{R}^{k+1}$ and $\mathbf{b}_k = \{b_j\}_{j=0}^k \in \mathbb{R}^{k+1}$ will be fixed later. Using standard arguments from the theory of ordinary differential equations, this system admits a unique solution $(\alpha_k, \beta_k) \subset (W^{1,2}(0, T_k))^{2(k+1)}$ on some interval $(0, T_k)$. For this solutions we have that either $|\alpha_k(t)| + |\beta_k(t)| \to \infty$ as $t \nearrow T_k$ or we can take any finite time T_k. In the following, it is shown that $|\alpha_k(t)| + |\beta_k(t)| \to \infty$ as $t \nearrow T_k$ does not occur by using a priori estimates. To this end, multiplying the first equation of (5.56) with $r \cdot \alpha_{kj}$, where r is the constant defined in Assumption 5.3.1, the second equation with β_{kj}, and summing over j yield

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(r\|u_k(t)\|_H^2 + \|w_k(t)\|_H^2\Big) + ra(u_k(t), u_k(t)) \\
&\quad + \int_\Omega rf(u_k(t), w_k(t))u_k(t) + g(u_k(t), w_k(t))w_k(t)\,\mathrm{d}x = r_{V'}\langle I(t), u_k(t)\rangle_V.
\end{aligned}$$

We recall that due to Lemma 5.1.2 the bidomain bilinear form a is coerciv and hence it is

$$\alpha\|U\|_V^2 \leq a(U, U) + \alpha\|U\|_H^2$$

for all $U \in V$ and for some constant $\alpha > 0$. By the coercivity of a, Assumption 5.3.1, and Young's inequality, it is

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(r\|u_k(t)\|_H^2 + \|w_k(t)\|_H^2\Big) + C_{11}\|u_k(t)\|_V^2 + C_{12}\|u_k(t)\|_p^p \\
&- C_{13}\|u_k(t)\|_H^2 + C_{14}\|w_k(t)\|_H^2 \leq C_{15}\|I(t)\|_{V'}^2 + C_{16},
\end{aligned} \tag{5.57}$$

for some constants $C_{1i} = C_{1i}(r, \alpha, C_j) > 0$ ($i = 1, \ldots, 6,\ j = 0, \ldots, 2$). Let us emphasize that all constants C_{1i} are independent of k. The assumption $2 \leq p < \infty$ yields the estimate

$$C_{17}\|u_k(t)\|_p^p - C_{18} \leq C_{12}\|u_k(t)\|_p^p - C_{13}\|u_k(t)\|_2^2$$

for some $C_{17}, C_{18} > 0$. Applying this estimate in (5.57), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(r\|u_k(t)\|_H^2 + \|w_k(t)\|_H^2\Big) \\
&+ C_{21}\Big(r\|u_k(t)\|_V^2 + \|u_k(t)\|_p^p + \|w_k(t)\|_H^2\Big) \leq C_{22}\|I(t)\|_{V'}^2 + C_{23},
\end{aligned} \tag{5.58}$$

for some constants $C_{2i} > 0$ $(i = 1, 2, 3)$.

Then, we use Gronwall's inequality and deduce from (5.58)

$$\begin{aligned} &r\|u_k(t)\|_H^2 + \|w_k(t)\|_H^2 \\ \le &\mathrm{e}^{-C_{21}t}(r\|\mathbf{a}_k\|_H^2 + \|\mathbf{b}_k\|_H^2) + \int_0^t \mathrm{e}^{-C_{21}(t-\tau)}(C_{22}\|I(\tau)\|_{V'}^2 + C_{23})\,\mathrm{d}\tau. \end{aligned} \tag{5.59}$$

Since $\|u_k(t)\|_H^2 = |\alpha_k(t)|^2$ and $\|w_k(t)\|_H^2 = |\beta_k(t)|^2$, this implies that $|\alpha_k(t)| + |\beta_k(t)|$ does not blow up at any finite time T_k.

Next, we take care of the periodicity. In order to do so, we consider the Poincaré map

$$\begin{aligned} \mathcal{S} : \mathbb{R}^{k+1} \times \mathbb{R}^{k+1} &\to \mathbb{R}^{k+1} \times \mathbb{R}^{k+1}, \\ \mathcal{S}(\mathbf{a}_k, \mathbf{b}_k) &:= (\alpha_k(T), \beta_k(T)), \end{aligned}$$

and define the ball

$$\mathbb{B}_R := \Bigg\{ (\mathbf{a}_k, \mathbf{b}_k) = (\{a_j\}_{j=0}^k, \{b_j\}_{j=0}^k) \in \mathbb{R}^{k+1} \times \mathbb{R}^{k+1} : \\ r\left(\sum_{j=0}^k |a_j|^2\right)^{1/2} + \left(\sum_{j=0}^k |b_j|^2\right)^{1/2} \le R \Bigg\}$$

with

$$R^2 = \frac{\int_0^T \mathrm{e}^{-C_{21}(T-\tau)}(C_{22}\|I(\tau)\|_{V'}^2 + C_{23})\,\mathrm{d}\tau}{1 - \mathrm{e}^{-C_{21}T}}. \tag{5.60}$$

Then, it follows that $\mathcal{S}$ maps $\mathbb{B}_R$ into itself from (5.59). Furthermore, $\mathcal{S}$ is also continuous. Hence we apply Brouwer's fixed point theorem to conclude that $\mathcal{S}$ admits a fixed point $(\bar{\mathbf{a}}_k, \bar{\mathbf{b}}_k) = \mathcal{S}(\bar{\mathbf{a}}_k, \bar{\mathbf{b}}_k)$ in $\mathbb{B}_R$ for all $k \in \mathbb{N}$.

In the following, we denote by u_k and w_k the functions defined in (5.54) and (5.55) respectively, corresponding to the solutions α_k, β_k of (5.56) with initial values $\bar{\mathbf{a}}_k$, $\bar{\mathbf{b}}_k$. Then, $u_k(0, x) = u_k(T, x)$ and $w_k(0, x) = w_k(T, x)$. Moreover, we see $u_k(t + T, x) = u_k(t, x)$ and $w_k(t + T, x) = w_k(t, x)$ for all $t \in \mathbb{R}$ by periodical expansion.

In the next step, we would like to pass to the limit $k \to \infty$ and show the existence of a weak solution to the original problem (PABDE). To do so, we consider the uniform boundedness. First, we take the supremum from $t = 0$ to $t = T$ in inequality (5.59) and use (5.60) to obtain

$$\|u_k\|_{L^\infty(0,T;H)} + \|w_k\|_{L^\infty(0,T;H)} \leq C_{31}\|I\|_{L^2(0,T;V')} + C_{32},$$

for some $C_{3i} > 0$ $(i = 1, 2)$. Moreover, we integrate the inequality (5.58) from $t = 0$ to $t = T$ to get

(5.61)
$$\|u_k\|^2_{L^2(0,T;V)} + \|u_k\|^p_{L^p(0,T;L^p(\Omega))} + \|w_k\|^2_{L^2(0,T;H)} \leq C_{41}\|I\|^2_{L^2(0,T;V')} + C_{42}.$$

for some $C_{4i} > 0$ $(i = 1, 2)$. This implies that there are sub-sequences of $\{u_k\}_{k=1}^\infty$ and $\{w_k\}_{k=1}^\infty$, for convenience still denoted by $\{u_k\}_{k=1}^\infty$ and $\{w_k\}_{k=1}^\infty$, that converges weakly to u in $L^2(0, T; V) \cap L^p(Q)$ and to w in $L^2(0, T; H)$, respectively.

By construction of the functions u_k and w_k we have

$$\begin{aligned}(\partial_t u_k(t), \psi_\ell) + a(u_k(t), \psi_\ell) + {}_{p'}\langle f(u_k, w_k), \psi_\ell\rangle_p &= {}_{V'}\langle I(t), \psi_\ell\rangle_V \\ (\partial_t w_k(t), \psi_\ell) + (g(u_k, w_k), \psi_\ell) &= 0\end{aligned}$$

for all $\ell = 0, \ldots, k$. Then, for some t_0, t_1 satisfying $0 \leq t_0 \leq t_1 \leq T$, we integrate from t_0 to t_1 to get

$$\begin{aligned}&|(u_k(t_1), \psi_\ell) - (u_k(t_0), \psi_\ell)| \\ =&|\int_{t_0}^{t_1} -a(u_k, \psi_\ell) - {}_{p'}\langle f(u_k, w_k), \psi_\ell\rangle_p + {}_{V'}\langle I, \psi_\ell\rangle_V \,\mathrm{d}\tau| \\ \leq&\mathrm{C}(M, \psi_\ell)\left(\int_{t_0}^{t_1} \|u_k\|_V + \|f(u_k, w_k)\|_{L^{p'}(\Omega)} \,\mathrm{d}\tau + \|I\|_{L^2(t_0,t_1;V')}\right) \\ \leq&C(M, \psi_\ell, I)\left(|t_1 - t_0|^{1/2} + |t_1 - t_0|^{1/p} + \|I\|_{L^2(t_0,t_1;V')}\right),\end{aligned}$$

as well as

$$\begin{aligned}&|(w_k(t_1), \psi_\ell) - (w_k(t_0), \psi_\ell)| \\ =&|\int_{t_0}^{t_1} -(g(u_k, w_k), \psi_\ell) \,\mathrm{d}\tau| \\ \leq&\|\psi_\ell\|_{L^2(\Omega)} \int_{t_0}^{t_1} \|g(u_k, w_k)\|_{L^2(\Omega)} \,\mathrm{d}\tau \\ \leq&C|t_1 - t_0|^{1/2}\end{aligned}$$

for some $C = C(I, \psi_\ell)$ which is independent of t_0, t_1 and k. Here we used inequality (5.61) and the embedding assumption $(\psi_\ell \in)\ V \hookrightarrow L^p(\Omega)$. Therefore, for $k = 1, 2, \ldots$ it follows that for any $\varepsilon > 0$ there exists a $\delta > 0$ with

$$|(u_k(t_1), \psi_\ell) - (u_k(t_0), \psi_\ell)| + |(w_k(t_1), \psi_\ell) - (w_k(t_0), \psi_\ell)| < \varepsilon,$$

if $|t_1 - t_0| \le \delta$. Hence, the families $\{(u_k(t), \psi_\ell)\}_{k=1}^\infty$ and $\{(w_k(t), \psi_\ell)\}_{k=1}^\infty$ are equicontinuous. Since $\{(u_k, \psi_\ell)\}_{k=1}^\infty$ and $\{(w_k, \psi_\ell)\}_{k=1}^\infty$ are uniformly bounded in k, we can apply the theorem of Arzela-Ascoli to conclude that the subsequences $\{(u_k(t), \psi_\ell)\}_{k=1}^\infty$ and $\{(w_k(t), \psi_\ell)\}_{k=1}^\infty$ converge uniformly to continuous functions $(u(t), \psi_\ell)$ and $(w(t), \psi_\ell)$ for each fixed ℓ. By the Cantor diagonalization argument and a density argument, this convergence can be generalized in a way that for each $\psi \in H$, the families $\{(u_k(t), \psi)\}_{k=1}^\infty$ and $\{(w_k(t), \psi)\}_{k=1}^\infty$ converge uniformly to continuous functions $(u(t), \psi)$ and $(w(t), \psi)$. Hence, we have $u \in C_w(0, T; H)$ and $w \in C_w(0, T; H)$.

Therefore, it remains to show the weak convergence of the nonlinear terms $f(u_k, w_k)$ and $g(u_k, w_k)$. We first prove $u_k \to u$ in $L^2(Q)$. To do so, we use Friedrich's inequality (see, e.g., [28, Lemma II.5.2]), which states that for any $\varepsilon > 0$, there exist $J \in \mathbb{N}$ and $\phi_1, \phi_2, \ldots, \phi_J \in H$ such that for all $U \in V$, the following inequality holds

$$\|U\|_H^2 \le \sum_{j=1}^{J} \left| \int_\Omega U \phi_j \, \mathrm{d}x \right|^2 + \varepsilon \|\nabla U\|_H^2.$$

This inequality with $U = u_k - u$, the uniform boundedness of $\{u_k\}_{k=1}^\infty \subset L^2(0, T; V)$, and $u_k \to u$ in $C_w(0, T; H)$ implies that $u_k \to u$ in $L^2(Q)$. Since we have $u_k \to u$ a.e. in Q and f_1, f_2, g_1 are continuous, $f_1(u_k) \to f_1(u)$, $f_2(u_k) \to f_2(u)$, $g_1(u_k) \to g_1(u)$ a.e. in Q are satisfied. Finally, we have to show uniform boundedness in $L^{p'}(Q)$ for $f(u_k, w_k)$ and uniform boundedness in $L^2(Q)$ for $g(u_k, w_k)$, which implies $f(u_k, w_k) \to f(u, w)$ weakly in $L^{p'}(Q)$ and $g(u_k, w_k) \to g(u, w)$ weakly in $L^2(Q)$. Fortunately, under the Assumption 5.3.1, this has already been proved in [13, p.477].

Since the functions u_k, w_k satisfy that for all $\varphi_1 \in W^{1,2}(0, T; H) \cap$

$L^2(0,T;V) \cap L^p(Q)$ and all $\varphi_2 \in W^{1,2}(0,T;H)$,

$$\int_0^t \{(u_k, \partial_t \varphi_1) - a(u_k, \varphi_1) - {}_{p'}\langle f(u_k, w_k), \varphi_1\rangle_p\} \,\mathrm{d}\tau$$
$$= -\int_0^t {}_{V'}\langle I, \varphi_1\rangle_V \,\mathrm{d}\tau + (u_k(t), \varphi_1(t)) - (u_k(0), \varphi_1(0)),$$
$$\int_0^t \{(w_k, \partial_t \varphi_2) - (g(u_k, w_k), \varphi_2)\} \,\mathrm{d}\tau = (w_k(t), \varphi_2(t)) - (w_k(0), \varphi_2(0)),$$

for all $t \in (0,T)$, combining above discussions about the weak convergence, we obtain the existence of a weak T-periodic solution to (PABDE). ☐

5.3.2 Regularity of Weak Periodic Solution

In this subsection, we show that in the case of FitzHugh–Nagumo type nonlinearities as introduced in Subsection 5.1.1 the weak time-periodic solution constructed in the previous subsection is actually a strong solution. In order to do so, we first review the global strong well-posedness result by Colli Franzone and Savaré [19]. After that, we use a weak-strong uniqueness argument to show the existence of a strong time-periodic solution for (PABDE) with FitzHugh–Nagumo nonlinearities.

Let $\mathcal{T}$ be a sufficiently large time such that $T < \mathcal{T}$. In [19], they considered the initial boundary value problem for the bidomain equations of the form

(BDE II)
$$\begin{cases} \partial_t u - \operatorname{div}(\sigma_i \nabla u_i) + F(u) + \theta w = I_i & \text{in } (0,\mathcal{T}) \times \Omega, \\ \partial_t u + \operatorname{div}(\sigma_e \nabla u_e) + F(u) + \theta w = -I_e & \text{in } (0,\mathcal{T}) \times \Omega, \\ u = u_i - u_e & \text{in } (0,\mathcal{T}) \times \Omega, \\ \partial_t w + \gamma w - \eta u = 0 & \text{in } (0,\mathcal{T}) \times \Omega, \\ \sigma_i \nabla u_i \cdot \nu = g_i \ \sigma_e \nabla u_e \cdot \nu = g_e, & \text{on } (0,\mathcal{T}) \times \partial\Omega, \\ u(0) = u_0, \ \ w(0) = w_0 & \text{in } \Omega, \end{cases}$$

with $\theta, \gamma, \eta > 0$.

They regarded the bidomain equation in a *degenerate* variational formulation and constructed a global weak formulation. For more details concerning their notation, we refer to [19].

Let Ω be a Lipschitz domain in $\mathbb{R}^d$, $\Gamma := \partial\Omega$, and the measurable functions $\sigma_{i,e} : \overline{\Omega} \to \mathbb{R}^{d\times d}$ satisfy the uniform ellipticity condition 5.1.1. Assume the nonlinear term F is a continuous function satisfying

(5.62)
$$F(0) = 0,\ \exists\lambda_F \geq 0 : \frac{F(x) - F(y)}{x - y} \geq -\lambda_F, \quad \forall x, y \in \mathbb{R},\ \text{with } x \neq y.$$

Then, their result reads as follows.

Theorem 5.3.4 (Franzone-Savaré '02 [19, Theorem 2]). *Assume $I_{i,e} \in L^2(0,\mathcal{T}; H)$, $g_{i,e} \in W^{1,1}(0,\mathcal{T}; H^{-1/2}(\Gamma))$ satisfy $I_i + I_e \in W^{1,1}(0,\mathcal{T}; H)$ and the compatibility condition*

$$\int_\Omega (I_i + I_e)\ \mathrm{d}x + {}_{H^{-1/2}(\Gamma)}\langle g_i + g_e, 1\rangle_{H^{1/2}(\Gamma)} = 0.$$

Then for any initial data $u_0, w_0 \in H$, there uniquely exist a couple

$$u_{i,e} \in L^2(0,\mathcal{T}; V),\ \int_\Omega u_e = 0 \text{ a.e } t$$

and

$$\begin{aligned}
&u \in C([0,\mathcal{T}]; H) \cap L^2(0,\mathcal{T}; V),\ \partial_t u \in L^2_{loc}(0,\mathcal{T}; H),\\
&F(u(t)) \in L^1(\Omega) \cap V' \ \text{a.e. } t \in (0,\mathcal{T}),\\
&w, \partial_t w \in C([0,\mathcal{T}]; H),
\end{aligned}$$

which solves the bidomain equation in the sense of

$$\begin{aligned}
&\int_\Omega (\partial_t u\hat{u} + \frac{\theta}{\eta}\partial_t w\hat{w})\ \mathrm{d}x + \int_\Omega F(u)\hat{u}\ \mathrm{d}x + \sum_{i,e}\int_\Omega \sigma_{i,e}\nabla u_{i,e}\cdot\nabla\hat{u}_{i,e}\ \mathrm{d}x\\
&+ \frac{\theta\gamma}{\eta}\int_\Omega w\hat{w}\ \mathrm{d}x + \theta\int_\Omega (w\hat{u} - u\hat{w})\ \mathrm{d}x\\
&= \sum_{i,e}\int_\Omega I_{i,e}\hat{u}_{i,e}\ \mathrm{d}x + \sum_{i,e} {}_{H^{-1/2}(\Gamma)}\langle g_{i,e}, \hat{u}_{i,e}\rangle_{H^{1/2}(\Gamma)},\\
&\int_\Omega (u(0)\hat{u} + \frac{\theta}{\eta}w(0)\hat{w})\ \mathrm{d}x = \int_\Omega (u_0\hat{u} + \frac{\theta}{\eta}w_0\hat{w})\ \mathrm{d}x,
\end{aligned}$$

for a.e. $t \in (0,\mathcal{T})$ and all $\hat{u}_{i,e} \in V \times V$ with $\int_\Omega \hat{u}_e\ \mathrm{d}x = 0$ and $\hat{u} = \hat{u}_i - \hat{u}_e$ and $\hat{w} \in H$.

Moreover if $u_0 \in V, u_0F(u_0) \in L^1(\Omega)$, then

$$u_{i,e} \in C([0,\mathcal{T}]; V),\ \partial_t u \in L^2(0,\mathcal{T}; H),\ w \in C([0,\mathcal{T}]; V).$$

Furthermore, they derived the following result implying higher regularity.

Proposition 5.3.5 ([19, Proposition 3.1])**.** *In addition to the assumptions in Theorem 5.3.4, suppose that* $d = 3$*, and the nonlinear term* F *has a cubic growth at infinity, i.e.,*

$$0 < \liminf_{|r|\to\infty} \frac{F(r)}{r^3} \leq \limsup_{|r|\to\infty} \frac{F(r)}{r^3} < +\infty.$$

Then the bidomain equation admits a unique strong solution $u_{i,e}, u, w$*. Moreover, it satisfies*

$$-\operatorname{div}(\sigma_{i,e}\nabla u_{i,e}) \in L^2(0, \mathcal{T}; H)$$

Remark 5.3.6 ([19, Remark 3.2])**.** Let Ω be of class $C^{1,1}$, $\sigma_{i,e}$ be Lipschitz in Ω and $g_{i,e} \in L^2(0, \mathcal{T}; H^{1/2}(\Gamma))$. Then by standard regularity estimates, we see

$$u_{i,e} \in L^2(0, \mathcal{T}; H^2(\Omega)).$$

Remark 5.3.7. If we look at the function f of the FitzHugh–Nagumo nonlinearity introduced in Section 5.1.1 as $f(u, w) = F(u) + w = u(u - a)(u - 1) + w$, then the function $F(u)$ satisfies the assumptions for the nonlinearity in Proposition 5.3.5 as well as (5.62).

After these preparations, we next combine the results from this section to obtain a strong time-periodic solution for the bidomain equations with FitzHugh–Nagumo type nonlinearities subject to arbitrary large forces.

In order to do so, we would like to identify our *weak* time-periodic solution (v, z) constructed in Subsection 5.3.1 with a *strong* solution (u, w) to the initial value problem. As initial data for this strong solution we choose $v(t_0), z(t_0)$ for some $t_0 > 0$ which satisfy $v(t_0) \in V$ and $f(v(t_0))v(t_0) \in L^1(\Omega)$. This choice of initial data is justified by the fact that for $p = 4$ the inequality $\|f(v)v\|_{L^1(Q)} \leq \|f(v)\|_{L^{p'}(Q)}\|v\|_{L^p(Q)}$ holds . This estimate guarantees the existence of a $t_0 > 0$ such that $f(v(t_0))(v(t_0)) \in L^1(\Omega)$. Hence, we are able to apply the theorem by Colli-Franzone and Savaré to obtain a global strong solution corresponding to the initial values $v(t_0)$,

$z(t_0)$. Finally, we show that the weak solution (v, z) coincides with the strong solution (u, w) and therefore obtain the existence of a strong time-periodic solution. We follow the approach given in [29].

To be more precise, for given T-periodic functions $I_{i,e} \in L^2(0, \mathcal{T}; H)$ with $I_i + I_e \in W^{1,1}(0, \mathcal{T}; H)$ and $\fint_\Omega (I_i + I_e)\, \mathrm{d}x = 0$ for a.e. t, let (v, z) be a weak T-periodic solution of (PABDE) corresponding to Theorem 5.3.3 where the external current is given by

$$I = I_i - A_i(A_i + A_e)^{-1}(I_i + I_e) \quad (\in L^2(0, T; H)).$$

We choose t_0 such that $v(t_0) \in V$ and $v(t_0) f(v(t_0)) \in L^1(\Omega)$. Since (v, z) is a weak T-periodic solution, for all $\varphi_1 \in W^{1,2}(t_0, \mathcal{T}; H) \cap L^2(t_0, \mathcal{T}; V) \cap L^4(Q)$ and all $\varphi_2 \in W^{1,2}(t_0, \mathcal{T}; H)$ it satisfies

$$\begin{aligned} &\int_{t_0}^t \{(v, \partial_t \varphi_1) - a(v, \varphi_1) - (f(v, z), \varphi_1)\}\, \mathrm{d}\tau \\ &\quad = -\int_{t_0}^t (I(\tau), \varphi_1(\tau))\, \mathrm{d}\tau + (v(t), \varphi_1(t)) - (v(t_0), \varphi_1(t_0)), \end{aligned} \tag{5.63}$$

$$\int_{t_0}^t \{(z, \partial_t \varphi_2) - (g(v, z), \varphi_2)\}\, \mathrm{d}\tau = (z(t), \varphi_2(t)) - (z(t_0), \varphi_2(t_0)), \tag{5.64}$$

for all $t \in (t_0, \mathcal{T})$. Furthermore, (v, z) satisfies the following strong energy inequality

$$\begin{aligned} &\left(\|v(t)\|_H^2 + \|z(t)\|_H^2\right) + 2\int_{t_0}^t a(v(\tau), v(\tau))\, \mathrm{d}\tau \\ &\quad + 2\int_{t_0}^t \int_\Omega f(v(\tau), z(\tau)) v(\tau) + g(v(\tau), z(\tau)) z(\tau)\, \mathrm{d}x\, \mathrm{d}\tau \\ &\le \|v(t_0)\|_H^2 + \|z(t_0)\|_H^2 + 2\int_{t_0}^t (I(\tau), v(\tau))\, \mathrm{d}\tau, \end{aligned} \tag{5.65}$$

for all $t \in [t_0, \mathcal{T}]$.

Next, let $(u, w) \in (W^{1,2}(t_0, \mathcal{T}; H) \cap L^2(t_0, \mathcal{T}; H^2(\Omega))) \times C^1([t_0, \mathcal{T}]; H)$ be the unique global strong solution corresponding to the initial-boundary value problem for the bidomain equations with initial value $(v(t_0), z(t_0))$, T-periodic right-hand side $I_{i,e}$, and $g_{i,e} = 0$. In the following, we show that the weak solution (v, z) agrees with the strong solution (u, w).

Since (u, w) is a strong solution, it satisfies that for all $\mathcal{T} > t_0$ and all $\phi_1 \in W^{1,2}(t_0, \mathcal{T}; H) \cap L^2(t_0, \mathcal{T}; V) \cap L^4(Q)$ and all $\phi_2 \in W^{1,2}(t_0, \mathcal{T}; H)$, we have

$$
(5.66) \quad \begin{aligned}
&\int_{t_0}^{t} \{(u, \partial_t \phi_1) - a(u, \phi_1) - (f(u,w), \phi_1)\} \, \mathrm{d}\tau \\
&\quad = -\int_{t_0}^{t} (I(\tau), \phi_1(\tau)) \, \mathrm{d}\tau + (u(t), \phi_1(t)) - (v(t_0), \phi_1(t_0)),
\end{aligned}
$$

$$
(5.67) \quad \int_{t_0}^{t} \{(w, \partial_t \phi_2) - (g(u,w), \phi_2)\} \, \mathrm{d}\tau = (w(t), \phi_2(t)) - (z(t_0), \phi_2(t_0)),
$$

for all $t \in (t_0, \mathcal{T})$. Furthermore, (u, w) satisfies the following strong energy identity

$$
(5.68) \quad \begin{aligned}
&\left(\|u(t)\|_H^2 + \|w(t)\|_H^2\right) + 2\int_{t_0}^{t} a(u(\tau), u(\tau)) \, \mathrm{d}\tau \\
&\quad + 2\int_{t_0}^{t} \int_{\Omega} f(u(\tau), w(\tau))u(\tau) + g(u(\tau), w(\tau))w(\tau) \, \mathrm{d}x \, \mathrm{d}\tau \\
&= \|v(t_0)\|_H^2 + \|z(t_0)\|_H^2 + 2\int_{t_0}^{t} (I(\tau), u(\tau)) \, \mathrm{d}\tau.
\end{aligned}
$$

Next, denote the mollified functions of v, z, u, and w by

$$
\begin{aligned}
v_h(t) &:= \int_0^{\mathcal{T}} j_h(t - \tilde{t}) v(\tilde{t}) \, \mathrm{d}\tilde{t}, \quad z_h(t) := \int_0^{\mathcal{T}} j_h(t - \tilde{t}) z(\tilde{t}) \, \mathrm{d}\tilde{t}, \\
u_h(t) &:= \int_0^{\mathcal{T}} j_h(t - \tilde{t}) u(\tilde{t}) \, \mathrm{d}\tilde{t}, \quad w_h(t) := \int_0^{\mathcal{T}} j_h(t - \tilde{t}) w(\tilde{t}) \, \mathrm{d}\tilde{t},
\end{aligned}
$$

respectively, where $j_h \in \mathrm{C}_c^\infty(-h, h)$, $0 < h < \mathcal{T}$, is even and positive with $\int_{\mathbb{R}} j_h(\tilde{t}) \, \mathrm{d}\tilde{t} = 1$. Then, as it is well known,

$$
\begin{aligned}
&\lim_{h \to 0} \int_0^{\mathcal{T}} \|v_h(\tau) - v(\tau)\|_V^2 \, \mathrm{d}\tau = 0, \\
&\operatorname{ess\,sup}_{t \in [0, \mathcal{T}]} \|v_h(t)\|_2 \leq \operatorname{ess\,sup}_{t \in [0, \mathcal{T}]} \|v(t)\|_2, \\
&\lim_{h \to 0} \int_0^{\mathcal{T}} \|u_h(\tau) - u(\tau)\|_{H^2}^2 \, \mathrm{d}\tau = 0, \\
&\operatorname{ess\,sup}_{t \in [0, \mathcal{T}]} \|u_h(t)\|_V \leq \operatorname{ess\,sup}_{t \in [0, \mathcal{T}]} \|u(t)\|_V, \\
&\lim_{h \to 0} \int_0^{\mathcal{T}} \|z_h(\tau) - z(\tau)\|_H^2 \, \mathrm{d}\tau = 0, \\
&\lim_{h \to 0} \int_0^{\mathcal{T}} \|w_h(\tau) - w(\tau)\|_H^2 \, \mathrm{d}\tau = 0.
\end{aligned}
$$

The weak continuity of v and u implies

$$\lim_{h\to 0}(u(t), v_h(t)) = \lim_{h\to 0}(u_h(t), v(t)) = (u(t), v(t)),\ t \geq t_0, \tag{5.69}$$

$$\lim_{h\to 0}(w(t), z_h(t)) = \lim_{h\to 0}(w_h(t), z(t)) = (w(t), z(t)),\ t \geq t_0. \tag{5.70}$$

Furthermore, integration by parts yields

$$\begin{aligned}
\int_{t_0}^{t}(u, \partial_t v_h)\,\mathrm{d}\tau &= -\int_{t_0}^{t}(\partial_t u, v_h)\,\mathrm{d}\tau + (u(t), v_h(t)) - (u(t_0), v_h(t_0)),\\
\int_{t_0}^{t}(w, \partial_t z_h)\,\mathrm{d}\tau &= -\int_{t_0}^{t}(\partial_t w, z_h)\,\mathrm{d}\tau + (w(t), z_h(t)) - (w(t_0), z_h(t_0)).
\end{aligned}$$

Then, by taking the limit we obtain

$$\begin{aligned}
\lim_{h\to 0}\left\{\int_{t_0}^{t}(u, \partial_t v_h) + (\partial_t u, v_h)\,\mathrm{d}\tau\right\} &= (u(t), v(t)) - \|v(t_0)\|_H^2\\
\lim_{h\to 0}\left\{\int_{t_0}^{t}(w, \partial_t z_h) + (\partial_t w, z_h)\,\mathrm{d}\tau\right\} &= (w(t), z(t)) - \|z(t_0)\|_H^2.
\end{aligned}$$

We now replace φ_1 by u_h in (5.63), φ_2 by w_h in (5.64), ϕ_1 by v_h in (5.66), and ϕ_2 by z_h in (5.67). Then, we sum up the resulting equations, pass to the limit $h \to 0$, and use the properties of the time-mollifier mentioned above to obtain

$$\begin{aligned}
&\int_{t_0}^{t}\{-2a(u, v) - (f(v, z), u) - (f(u, w), v) - (g(v, z), w)\\
&- (g(u, w), z)\}\,\mathrm{d}\tau = -\int_{t_0}^{t}(I(\tau), u(\tau) + v(\tau))\,\mathrm{d}\tau\\
&+ (u(t), v(t)) - \|v(t_0)\|_H^2 + (w(t), z(t)) - \|z(t_0)\|_H^2.
\end{aligned} \tag{5.71}$$

Now we have all ingredients to prove $(u, w) = (v, z)$. In order to do so, we calculate

$$\begin{aligned}
&\|u(t) - v(t)\|_H^2 + \|w(t) - z(t)\|_H^2 + 2\int_{t_0}^{t} a(u(\tau) - v(\tau), u(\tau) - v(\tau))\,\mathrm{d}\tau\\
&= \left(\|u(t)\|_H^2 + \|w(t)\|_H^2 + 2\int_{t_0}^{t} a(u(\tau), u(\tau))\,\mathrm{d}\tau\right)\\
&\quad + \left(\|v(t)\|_H^2 + \|z(t)\|_H^2 + 2\int_{t_0}^{t} a(v(\tau), v(\tau))\,\mathrm{d}\tau\right)\\
&\quad - 2(u(t), v(t)) - 2(w(t), z(t)) - 4\int_{t_0}^{t} a(u(\tau), v(\tau))\,\mathrm{d}\tau.
\end{aligned}$$

For the first two parts on the right-hand side of the equation, we use the strong energy equality (5.68) and the strong energy inequality (5.65), and for the last term, we use the relation (5.71). Then, we have

$$\begin{aligned}
&\|u(t)-v(t)\|_H^2+\|w(t)-z(t)\|_H^2+2\int_{t_0}^{t} a(u(\tau)-v(\tau),u(\tau)-v(\tau))\,\mathrm{d}\tau\\
&\le 2\int_{t_0}^{t}\{(f(v,z),u)+(f(u,w),v)-(f(u,w),u)-(f(v,z),v)\\
&\qquad +(g(v,z),w)+(g(u,w),z)-(g(u,w),w)-(g(v,z),z)\}\,\mathrm{d}\tau\\
&\le -2\int_{t_0}^{t}(f(u,w)-f(v,z),u-v)+(g(u,w)-g(v,z),w-z)\,\mathrm{d}\tau
\end{aligned}$$

We estimate the parts on the right-hand side separately. For the first term we use (5.62) and Young's inequality to get

$$\begin{aligned}
&-2\int_{t_0}^{t}(f(u,w)-f(v,z),u-v)\,\mathrm{d}\tau\\
&\le 2\lambda_f\int_{t_0}^{t}\|u(\tau)-v(\tau)\|_H^2\,\mathrm{d}\tau-2\int_{t_0}^{t}(w-z,u-v)\,\mathrm{d}\tau\\
&\le 2\lambda_f\int_{t_0}^{t}\|u(\tau)-v(\tau)\|_H^2\,\mathrm{d}\tau+\int_{t_0}^{t}\varepsilon_1\|w-z\|_H^2+C(\varepsilon_1)\|u-v\|_H^2\,\mathrm{d}\tau
\end{aligned}$$

for some constants ε_1, $C(\varepsilon_1)>0$. For the second term we use that the function $g(u,w)=-\varepsilon(ku-w)$ is linear, hence

$$|(g(u,w)-g(v,z),w-z)|\le C(\|u-v\|_H^2+\|w-z\|_H^2).$$

for some $C>0$. Therefore, combining these estimates yields

$$\begin{aligned}
&\|u(t)-v(t)\|_H^2+\|w(t)-z(t)\|_H^2+2\int_{t_0}^{t} a(u(\tau)-v(\tau),u(\tau)-v(\tau))\,\mathrm{d}\tau\\
&\le C\int_{t_0}^{t}(\|u(\tau)-v(\tau)\|_H^2+\|w(\tau)-z(\tau)\|_H^2\,\mathrm{d}\tau,
\end{aligned}$$

for some $C>0$. Hence, we are able to apply Gronwall's lemma to conclude that

$$u-v\equiv 0, w-z\equiv 0 \text{ a.e. in } \Omega\times[t_0,\mathcal{T}].$$

This implies the existence of a strong T-periodic solution (u, w) when the source term $I_{i,e}$ is a T-periodic function.

Then, the main theorem on existence of strong time-periodic solutions without assuming smallness conditions for the external forces reads as follows.

Theorem 5.3.8. *Let $d = 3$, $T > 0$, and $I_{i,e} \in L^2(0, T; H)$ be T-periodic with $I_i + I_e \in W^{1,1}(0, T; H)$ and $\int_\Omega (I_i + I_e)\, \mathrm{d}x = 0$ for a.e. t. Let the conductivity matrices $\sigma_{i,e}$ satisfy Assumption 5.1.1 and the nonlinear term F satisfy (5.62) and assume that there exist constants $C_0 \in \mathbb{R}$ and $C_1 > 1$ such that*

$$C_0 + C_1|u|^4 \leq F(u)u \tag{5.72}$$

for all $u \in \mathbb{R}$. Then for the bidomain equations with FitzHugh–Nagumo type transport

$$\begin{cases} \partial_t u - \operatorname{div}(\sigma_i \nabla u_i) + F(u) + w = I_i, & \text{in } \mathbb{R} \times \Omega, \\ \partial_t u + \operatorname{div}(\sigma_e \nabla u_e) + F(u) + w = -I_e, & \text{in } \mathbb{R} \times \Omega, \\ u = u_i - u_e, & \text{in } \mathbb{R} \times \Omega, \\ \partial_t w - \varepsilon(ku - w) = 0, & \text{in } \mathbb{R} \times \Omega, \\ \sigma_i \nabla u_i \cdot \nu = 0,\ \sigma_e \nabla u_e \cdot \nu = 0, & \text{on } \mathbb{R} \times \partial\Omega, \\ u(t + T, x) = u(t, x), & \text{in } \mathbb{R} \times \Omega, \\ w(t + T, x) = w(t, x), & \text{in } \mathbb{R} \times \Omega, \end{cases}$$

there exists a strong time-periodic solution

$$\begin{aligned} &(u_i, u_e) \in (W^{1,2}(0, T; H) \cap L^2(0, T; H^2(\Omega)))^2 \ \textit{with} \int_\Omega u_e\, \mathrm{d}x = 0 \text{ a.e. } t \\ &(u, w) \in (W^{1,2}(0, T; H) \cap L^2(0, T; H^2(\Omega)) \cap L^4(Q)) \times C^1(0, T; H). \end{aligned}$$

Remark 5.3.9. The Assumption 5.3.1 of the existence of the weak periodic solutions is replaced by (5.72).

Remark 5.3.10. We do not treat the ionic models by Rogers–McCulloch and Aliev-Panfilov due to the lack of a suitable global well-posedness result for the initial value problem in the L^2-setting.

5.3.3 Nonlinearities

In this subsection, we check that the three ionic models mentioned at the beginning of Subsection 5.3.1 satisfy Assumption 5.3.1. Since the growth conditions (5.51)-(5.53) are trivial with $p = 4$, we only show condition (5.50).

FitzHugh–Nagumo model

The FitzHugh–Nagumo type ionic model is

$$\begin{aligned} f(u,w) &= u(u-a)(u-1)+w \\ g(u,w) &= -\varepsilon(ku-w) \end{aligned}$$

with $0 < a < 1$ and $k, \varepsilon > 0$. Then, we are able to calculate as follows ($r = 1$):

$$f(u,w)u + g(u,w)w = u^4 - (a+1)u^3 + au^2 + uw - \varepsilon kuw + \varepsilon w^2$$

and by

$$\begin{aligned} |(a+1)u^3| &\le \frac{1}{8}u^4 + c_{11}, \\ |au^2| &\le \frac{1}{8}u^4 + c_{12}, \\ |uw| &\le \frac{1}{8}u^4 + \frac{\varepsilon}{4}w^2 + c_{13}, \\ |\varepsilon uw| &\le \frac{1}{8}u^4 + \frac{\varepsilon}{4}w^2 + c_{14}, \end{aligned}$$

for some $c_{1i} > 0$ ($i = 1, \dots, 4$), we have

$$f(u,w)u + g(u,w)w \ge \frac{1}{2}u^4 + \frac{\varepsilon}{2}w^2 + c_1$$

for some $c_1 \in \mathbb{R}$. Therefore, the FitzHugh–Nagumo model satisfies Assumption 5.3.1.

Rogers–McCulloch model

The Rogers–McCulloch type ionic model is

$$\begin{aligned} f(u,w) &= bu(u-a)(u-1) + uw \\ g(u,w) &= -\varepsilon(ku - w) \end{aligned}$$

with $0 < a < 1$ and $b, k, \varepsilon > 0$. Then, we are able to calculate as follows:

$$\begin{aligned} &rf(u,w)u + g(u,w)w \\ &\qquad = rbu^4 - rb(a+1)u^3 + rbau^2 + ru^2w - \varepsilon kuw + \varepsilon w^2 \end{aligned}$$

and, based on the calculation

$$|ru^2w| \leq \frac{C^2}{2}u^4 + \frac{r^2}{2C^2}w^2,$$

we choose $r, C > 0$ depending on b, ε, such that

$$\begin{cases} c_{21} := rb - \frac{C^2}{2} > 0, \\ c_{22} := \varepsilon - \frac{r^2}{2C^2} > 0. \end{cases}$$

By

$$\begin{aligned} |rb(a+1)u^3| &\leq \frac{c_{21}}{6}u^4 + c_{23}, \\ |rbau^2| &\leq \frac{c_{21}}{6}u^4 + c_{24}, \\ |\varepsilon kuw| &\leq \frac{c_{21}}{6}u^4 + \frac{c_{22}}{2}w^2 + c_{25}, \end{aligned}$$

for some $c_{2i} > 0$ $(i = 3, \ldots, 5)$, we have

$$rf(u,w)u + g(u,w)w \geq \frac{c_{21}}{2}u^4 + \frac{c_{22}}{2}w^2 + c_2$$

for some $c_2 \in \mathbb{R}$. Therefore, the Rogers–McCulloch model satisfies Assumption 5.3.1.

Aliev–Panfilov model

The modified Aliev–Panfilov type ionic model is

$$\begin{aligned} f(u,w) &= bu(u-a)(u-1) + uw \\ g(u,w) &= \varepsilon(ku(u-1-d) + w) \end{aligned}$$

with $0 < a, d < 1$, $b, k, \varepsilon > 0$, and $b > k$ (in the original Aliev–Panfilov model we have $b = k$). Then, we are able to calculate as follows:

$$\begin{aligned} &rf(u,w)u + g(u,w)w \\ &\quad = rbu^4 + rb(a+1)u^3 + rbau^2 + ru^2w + \varepsilon ku^2w - \varepsilon k(1+d)uw + \varepsilon w^2 \end{aligned}$$

and, based on the calculation

$$|(r+\varepsilon k)u^2w| \leq \frac{C^2}{2}u^4 + \frac{(r+\varepsilon k)^2}{2C^2}w^2,$$

we choose $r, C > 0$ depending on b, k, ε, such that

$$\begin{cases} c_{31} := rb - \frac{C^2}{2} > 0, \\ c_{32} := \varepsilon - \frac{(r+\varepsilon k)^2}{2C^2} > 0. \end{cases}$$

Here, the assumption $b > k$ is essential. By

$$\begin{aligned} |rb(a+1)u^3| &\leq \frac{c_{31}}{6}u^4 + c_{33}, \\ |rbau^2| &\leq \frac{c_{31}}{6}u^4 + c_{34}, \\ |\varepsilon k(1+d)uw| &\leq \frac{c_{31}}{6}u^4 + \frac{c_{32}}{2}w^2 + c_{35}, \end{aligned}$$

for some $c_{3i} > 0$ $(i = 3, \ldots, 5)$, we have

$$rf(u,w)u + g(u,w)w \geq \frac{c_{31}}{2}u^4 + \frac{c_{32}}{2}w^2 + c_3$$

for some $c_3 \in \mathbb{R}$. Note that we are not able to choose a suitable c_{3i} $(i = 1, 2)$ in the case $b = k$. Therefore, the modified Aliev-Panfilov model satisfies Assumption 5.3.1.

Liquid Crystals

CHAPTER 6

Time-Periodic Solutions to the Q-Tensor Model

In this chapter, we consider the Beris–Edwards Q-tensor model of nematic liquid crystals. We show that provided the system is innervated by a time-periodic external forcing, the system has a unique time-periodic solution of the same period.

This chapter is structured as follows. First, we briefly introduce the Beris–Edwards model and the Q-tensor. Then, we give an abstract quasilinear formulation for the model in order to combine the results from Subsection 2.5.3 with the recent results given in [84, Chapter 3] to obtain time-periodic solutions for this system. Finally, in Section 6.3 we consider some modified Beris–Edwards models.

6.1 The Beris–Edwards Model and the Q-Tensor

The Beris–Edwards model and the corresponding Q-tensor describing the molecular orientation of the liquid crystal flow was introduced by Beris and Edwards [11], and by de Gennes and Prost [32], respectively. For a detailed description of the model see, e.g., [84].

First, let us briefly introduce the Q-tensor. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded C^3-domain representing the underlying liquid crystal material. Let $x \in \Omega$ be a certain spatial point and ρ_x be the probability density function of molecular orientations. These orientations are assumed to lie in the unit sphere in $\mathbb{R}^d$ which denoted by $\mathcal{S}^{d-1}$. The surface measure of this sphere is denoted by σ^{d-1}. Then, the Q-tensor is defined as the traceless second-order moment of the probability function ρ_x

$$Q(x) := \int_{\mathcal{S}^{d-1}} \left(\omega \otimes \omega - \frac{1}{d}\mathbb{I} \right) \rho_x(\omega) \mathrm{d}\sigma^{d-1}(\omega), \quad x \in \Omega. \tag{6.1}$$

The Q-tensor is traceless, since for $\omega \in \mathcal{S}^{d-1}$ it is $\operatorname{tr}(\omega \otimes \omega) = |\omega|^2 = 1$. Hence, the Q-tensor is an element of the space of symmetric, traceless matrices

$$Q(x) \in \mathbb{S}^d_{0,\mathbb{R}} := \{Q \in \mathbb{R}^{d\times d} : Q = Q^T, \operatorname{tr} Q = 0\}.$$

Let $D := Du := \frac{1}{2}(\nabla u + (\nabla u)^T)$ and $W := Wu := \frac{1}{2}(\nabla u - (\nabla u)^T)$ be the symmetric and anti-symmetric part of the gradient of the velocity, respectively. Furthermore, define

$$\begin{aligned}
S &= (\xi D + W)(Q + \mathbb{I}/d) + (Q + \mathbb{I}/d)(\xi D - W) - 2\xi(Q + \mathbb{I}/d)\operatorname{tr}(Q\nabla u),\\
H &= \lambda \Delta Q - aQ + b(Q^2 - \operatorname{tr}(Q^2)\mathbb{I}/d) - c\operatorname{tr}(Q^2)Q,\\
\tau &= 2\xi(Q + \mathbb{I}/d)\operatorname{tr}(QH) - \lambda \nabla Q \odot \nabla Q - \xi(Q + \mathbb{I}/d)H - \xi H(Q + \mathbb{I}/d),\\
\sigma &= \lambda(Q\Delta Q - \Delta Q Q).
\end{aligned}$$

Here, the (i, j)-th component of $\nabla Q \odot \nabla Q$ equals $\operatorname{tr}(\partial_i Q \partial_j Q)$. Then, the

periodic Beris–Edwards model is given by
(BE)

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = \operatorname{div}(\tau(Q) + \sigma(Q)) + g_1(t) & \text{in } \mathbb{R} \times \Omega, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R} \times \Omega, \\ \partial_t Q + (u \cdot \nabla)Q - S(\nabla u, Q) = \Gamma H(Q) + g_2(t) & \text{in } \mathbb{R} \times \Omega, \\ (u, \partial_{\vec{\nu}} Q) = (0,0) & \text{on } \mathbb{R} \times \partial\Omega, \\ (u(0), Q(0)) = (u(T), Q(T)) & \text{in } \Omega. \end{cases}$$

The unknowns of this system are the velocity $u\colon \mathbb{R} \times \Omega \to \mathbb{R}^d$, the pressure $p\colon \mathbb{R}\times\Omega \to \mathbb{R}$, and the Q-tensor $Q\colon \mathbb{R}\times\Omega \to \mathbb{S}^d_{0,\mathbb{R}}$ which describes the molecular orientation. The external forces $g_1(t)$, $g_2(t)$ are assumed to be T-periodic for some $T > 0$. The parameter $\xi \in \mathbb{R}$ describes the ratio of tumbling and alignment effects whereas $\nu > 0$ is the viscosity constant.

For simplicity, in the following the constants are set as $\nu = \Gamma = \lambda = a = b = c = 1$, which does not change our analysis.

6.2 Time-Periodic Solutions

In this section, we prove the existence of time-periodic solutions to the Beris–Edwards model (BE) provided the system is innervated by periodic external forces g_1 and g_2.

Therefore, we start by rewriting the Beris–Edwards model as an abstract quasilinear evolution equation for which we can apply the abstract theory described in detail in Subsection 2.5.3.

To this end, recall that $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded C^3-domain. For $q \in (1,\infty)$, let $\mathbb{P}\colon \mathrm{L}^q(\Omega) \to \mathrm{L}^q_\sigma(\Omega)$ be the Helmholtz projection in $\mathrm{L}^q(\Omega)$. Then, we denote by $A_D := \mathbb{P}\Delta_D$ the Stokes operator on $L^q_\sigma(\Omega)$ with domain

$$D(A_D) := \{u \in W^{2,q}(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega\}.$$

Next, for the Q-tensor equation, we define the shifted Neumann-Laplacian D_N on $W^{1,q}(\Omega, \mathbb{S}^d_{0,\mathbb{C}})$ by $\mathrm{D}_N := \Delta - \mathbb{I}$ with domain

$$D(D_N) := \{Q \in W^{3,q}(\Omega; \mathbb{S}^d_{0,\mathbb{C}}) : \partial_{\vec{\nu}} Q = 0 \text{ on } \partial\Omega\}.$$

Here the space $\mathbb{S}^n_{0,\mathbb{C}}$ is equipped with the norm $\|Q\|^2_{\mathbb{S}^n_{0,\mathbb{C}}} = \operatorname{tr}(QQ^*)$, where Q^* denotes the conjugate transpose of Q.

After these preparations, we can now define the following solution and data spaces

$$
\begin{aligned}
X_0 &:= L^q_\sigma(\Omega) \times W^{1,q}(\Omega; \mathbb{S}^d_{0,\mathbb{C}}) \\
\mathbb{E}_1 &:= L^p(0,T; D(A_D)) \cap W^{1,p}(0,T; L^q_\sigma(\Omega)) \\
\mathbb{E}_2 &:= L^p(0,T; D(D_N)) \cap W^{1,p}(0,T; W^{1,q}(\Omega; \mathbb{S}^d_{0,\mathbb{C}}))
\end{aligned}
\tag{6.2}
$$

as well as

$$
\mathbb{F} := L^p(0,T; X_0) \quad \text{and} \quad \mathbb{E} := \mathbb{E}_1 \times \mathbb{E}_2. \tag{6.3}
$$

Let $v = (u, Q)^T$. Following [84, Subsection 3.1.5], we define on X_0 the operator $\mathcal{A}_\xi$ as

$$
\mathcal{A}_\xi(v) := \begin{pmatrix} A_D & -\mathbb{P} \operatorname{div} S_\xi(Q)(-\Delta + \mathbb{I}) \\ -\tilde{S}_\xi(Q)\nabla & D_N \end{pmatrix}. \tag{6.4}
$$

For the precise definitions of S_ξ and $\tilde{S}_\xi$ see [84, Definition 3.1.1]. Furthermore, for $v = (u, Q)^T$ we define the nonlinear term $F(t, v(t)) = (\mathbb{P} f_1(v) + g_1(t), f_2(v) + g_2(t))$ by

$$
\begin{cases}
f_1(v) = & -(u \cdot \nabla)u - \operatorname{div}(\nabla Q \odot \nabla Q) \\
& +2\xi \operatorname{div}\Big((Q + \mathbb{I}/d)(\operatorname{tr}(Q^3) - \operatorname{tr}(Q^2)^2)\Big) \\
& -2\xi \operatorname{div}\Big((Q + \mathbb{I}/d)(Q^2 - \operatorname{tr}(Q^2)\mathbb{I}/d - \operatorname{tr}(Q^2)Q)\Big), \\
f_2(v) = & (Q^2 - \operatorname{tr}(Q^2)\mathbb{I}/d - \operatorname{tr}(Q^2)Q - (u \cdot \nabla)Q.
\end{cases}
\tag{6.5}
$$

Using this notation, the Beris–Edwards Q-tensor model of nematic liquid crystals (BE) corresponds to the equation

$$
\begin{cases}
\partial_t v(t) - \mathcal{A}_\xi(v(t))v(t) = F(t, v(t)) \quad t \in (0,T), \\
v(0) = v(T).
\end{cases}
\tag{6.6}
$$

Then, the theorem on existence and uniqueness of strong time-periodic solutions to (BE) reads as follows.

Theorem 6.2.1. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded C^3-domain, $p > \frac{4}{4-d}$, $q = 2$, and $\xi \in \mathbb{R}$. Let $T > 0$ and $g = (g_1, g_2)^T \in \mathbb{F}$ be T-periodic.*

Then there is $r_0 > 0$ such that for any $r \in (0, r_0)$ there exists $\delta = \delta(r) > 0$ such that if $\|g\|_{\mathbb{F}} < \delta$, there exists a T-periodic solution $v = (u, Q)^T \in \mathbb{E}$ to (BE)*, which is unique in $\overline{B_{\mathbb{E}}}(0, r)$.*

Proof. In order to prove the theorem, by Proposition 2.5.6 it suffices to show the following properties:

a) The operator $\mathcal{A}_\xi(0)$ admits maximal periodic L^p-regularity on X_0.

b) There exists $R_0 > 0$ such that for each $R \in (0, R_0)$ there exists $L(R) > 0$ such that

$$\|\mathcal{A}_\xi(v(\cdot))z(\cdot) - \mathcal{A}_\xi(\tilde{v}(\cdot))z(\cdot)\|_{\mathbb{F}} \le L(R)\|v - \tilde{v}\|_{\mathbb{E}}\|z\|_{\mathbb{E}}$$

for all $v, \tilde{v}, z \in \overline{B_{\mathbb{E}}}(0, R)$.

c) $F(\cdot, v(\cdot)) \in \mathbb{F}$ for any $v \in \mathbb{E}$.

d) There exists a $C > 0$ such that for any $R > 0$ it is

$$\|F(\cdot, v(\cdot)) - F(\cdot, \tilde{v}(\cdot))\|_{\mathbb{F}} \le CR\|v - \tilde{v}\|_{\mathbb{E}}$$

for all $v, \tilde{v} \in \overline{B_{\mathbb{E}}}(0, R)$.

Due to [84, Theorem 3.2.16] the operator $\mathcal{A}_\xi(0)$ has the property of maximal L^p-regularity for $p > \frac{4}{4-d}$, $q = 2$. Furthermore, by [84, Proposition 3.2.14] the operator is invertible. Hence, Proposition 2.2.1 yields that $\mathcal{A}_\xi(0)$ admits maximal periodic L^p-regularity on X_0, which proves a).

By [84, Proposition 3.2.3] $\mathcal{A}_\xi$ is locally Lipschitz continuous. Note that $p > \frac{4}{4-d}$ and $q = 2$ imply the conditions on p and q assumed therein. This proves b).

Next, we verify assertion c). Considering that, [84, Proposition 3.4.4] states, that the nonlinear term F satisfies the assumptions of [84, Theorem 2.5.1]. Hence, $F(\cdot, v(\cdot)) \in \mathbb{F}$ is satisfied for $v \in \mathbb{E}$. Furthermore, for some $k \in \mathbb{N}_0$ we have

$$\begin{aligned}&\|F(\cdot, v(\cdot)) - F(\cdot, \tilde{v}(\cdot))\|_{\mathbb{F}} \\ &\quad \le C(\|v\|_{L^\infty(0,T;X_\gamma)} + \|\tilde{v}\|_{L^\infty(0,T;X_\gamma)} + 1)^k \cdot (\|v\|_{\mathbb{E}} + \|\overline{v}\|_{\mathbb{E}})\|v - \tilde{v}\|_{\mathbb{E}},\end{aligned}$$

for all $v, \tilde{v} \in \mathbb{E}$. Recall that for $X_1 = D(A_D) \times D(D_N)$ the trace spare X_γ is defined by $X_\gamma := (X_0, X_1)_{1-1/p,p}$ and that the following embedding is valid (see, e.g., [4, Theorem 4.10.2 in Chapter III])

$$\mathbb{E} \hookrightarrow BUC([0,T]; X_\gamma).$$

Hence, we obtain

$$\|F(\cdot, v(\cdot)) - F(\cdot, \tilde{v}(\cdot))\|_{\mathbb{F}} \le C(\|v\|_{\mathbb{E}} + \|\tilde{v}\|_{\mathbb{E}} + 1)^k \cdot (\|v\|_{\mathbb{E}} + \|\tilde{v}\|_{\mathbb{E}})\|v - \tilde{v}\|_{\mathbb{E}},$$

Then, for $v, \tilde{v} \in \overline{B_{\mathbb{E}}}(0, R)$ this yields

$$\begin{aligned}\|F(\cdot, v(\cdot)) - F(\cdot, \tilde{v}(\cdot))\|_{\mathbb{F}} &\le C(2R+1)^k \cdot (2R)\|v - \tilde{v}\|_{\mathbb{E}} \\ &\le CR^{k+1}\|v - \tilde{v}\|_{\mathbb{E}}.\end{aligned}$$

This proves d) and hence, the proof is complete. □

6.3 Time-Periodic Solutions for Modified Models

In this section, we consider time-periodic solutions for some modified Beris–Edwards models. For these modified models it was proved in [84, Chapter 3] that the involved operator matrices admit maximal regularity also for cases different from $q = 2$.

The first modification we consider is that the term S defined below equation (BE) is set to zero. Then, the operator $\mathcal{A}_\xi(v)$ defined in (6.4) reduces to an upper triangular matrix of the form

$$\tilde{\mathcal{A}}_\xi(v) := \begin{pmatrix} A_D & -\mathbb{P}\operatorname{div} S_\xi(\hat{Q})(-\Delta + \mathbb{I}) \\ 0 & D_N \end{pmatrix}. \tag{6.7}$$

with $S_\xi(\hat{Q})\colon \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d}$ defined by

$$S_\xi(\hat{Q})Q = [\hat{Q}, Q] - \xi\left(2Q/d + \{\hat{Q}, Q\} - 2(\hat{Q} + \mathbb{I}/d)\operatorname{tr}(\hat{Q}Q)\right).$$

Then, we consider (6.6) with $\mathcal{A}_\xi$ replaced by $\tilde{\mathcal{A}}_\xi(v)$. Provided that

$$p, q \in (1, \infty) \quad \text{such that} \quad \frac{2}{3p} + \frac{d}{3q} < 1 \tag{6.8}$$

the result on existence and uniqueness of the modified model reads as.

Theorem 6.3.1. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded C^3-domain, $\xi \in \mathbb{R}$ and assume that* (6.8) *is satisfied. Let $T > 0$ and $g = (g_1, g_2)^T \in \mathbb{F}$ be T-periodic and assume that the term S equals* 0*.*

Then there is $r_0 > 0$ such that for any $r \in (0, r_0)$ there exists $\delta = \delta(r) > 0$ such that if $\|g\|_{\mathbb{F}} < \delta$, there exists a T-periodic solution $v = (u, Q)^T \in \mathbb{E}$ to (BE)*, which is unique in $\overline{B_{\mathbb{E}}}(0, r)$.*

Proof. In order to prove the theorem, by Proposition 2.5.6 it suffices to show the following properties:

a) The operator $\tilde{\mathcal{A}}_\xi(0)$ admits maximal periodic L^p-regularity on X_0.

b) There exists $R_0 > 0$ such that for each $R \in (0, R_0)$ there exists $L(R) > 0$ such that

$$\|\tilde{\mathcal{A}}_\xi(v(\cdot))z(\cdot) - \tilde{\mathcal{A}}_\xi(\tilde{v}(\cdot))z(\cdot)\|_{\mathbb{F}} \leq L(R)\|v - \tilde{v}\|_{\mathbb{E}}\|z\|_{\mathbb{E}}$$

for all $v, \tilde{v}, z \in \overline{B_{\mathbb{E}}}(0, R)$.

c) $F(\cdot, v(\cdot)) \in \mathbb{F}$ for any $v \in \mathbb{E}$.

d) There exists a $C > 0$ such that for any $R > 0$ it holds

$$\|F(\cdot, v(\cdot)) - F(\cdot, \tilde{v}(\cdot))\|_{\mathbb{F}} \leq CR\|v - \tilde{v}\|_{\mathbb{E}}$$

for all $v, \tilde{v} \in \overline{B_{\mathbb{E}}}(0, R)$.

Since the nonlinearity F is the same as in the proof of Theorem 6.2.1, assertions c) and d) are already proved. So it remains to prove a) and b).

Due to [84, Proposition 3.3.2] the operator $\tilde{\mathcal{A}}_\xi(0)$ has the property of maximal regularity provided (6.8) is valid. Furthermore, by [84, Proposition 3.5.1] the operator is also invertible in this case. Hence, Proposition 2.2.1 yields that $\tilde{\mathcal{A}}_\xi(0)$ admits maximal periodic L^p-regularity on X_0, which proves a).

Finally, [84, Proposition 3.3.2] also states that $\tilde{\mathcal{A}}_\xi$ is locally Lipschitz continuous which proves b) and hence finishes the proof. □

Next, we consider (BE) with a modified stress tensor. More precisely, we replace the symmetric part of the stress tensor τ by

$$\tau_{\text{mod}}(Q, H) := \tau(Q, H) + 2\xi H(Q + \mathbb{I}/d).$$

Note that τ_{mod} is no longer symmetric and contains no linear parts

$$\tau_{\text{mod}} = 2\xi(Q + \mathbb{I}/d)\operatorname{tr}(QH) - \nabla Q \odot \nabla Q - \xi QH + \xi HQ.$$

Next, let $v = (u, Q)^T$. Following [84, Subsection 3.3.3], we define on X_0 the operator $\mathcal{A}_{\text{mod}}$ as

$$\mathcal{A}_{\text{mod}}(v) := \begin{pmatrix} A_D & -\mathbb{P}\operatorname{div} S_{\text{mod}}(Q)(-\Delta + \mathbb{I}) \\ -\tilde{S}_\xi(Q)\nabla & D_N \end{pmatrix}, \tag{6.9}$$

with $\tilde{S}_\xi$ as before and

$$S_{\text{mod}}(\hat{Q})Q := [\hat{Q}, Q] - \xi\left([\hat{Q}, Q] - 2(\hat{Q} + \mathbb{I}/d)\operatorname{tr}(\hat{Q}Q)\right).$$

Furthermore, for $v = (u, Q)^T$ we define the nonlinear term $F_{\text{mod}}(t, v(t)) = (\mathbb{P}f_{1,\text{mod}}(v) + g_1(t), f_2(v) + g_2(t))$ by

$$\begin{aligned} f_{1,\text{mod}}(v) = &- (u \cdot \nabla)u - \operatorname{div}(\nabla Q \odot \nabla Q) \\ &+ 2\xi \operatorname{div}\left((Q + \mathbb{I}/d)(\operatorname{tr}(Q^3) - \operatorname{tr}(Q^2)^2)\right), \end{aligned}$$

and f_2 as before. Then, the Beris–Edwards model (BE) with modified stress tensor τ_{mod} can be rewritten as a quasilinear evolution equation of the form

$$\begin{cases} \partial_t v(t) - \mathcal{A}_{\text{mod}}(v(t))v(t) = F_{\text{mod}}(t, v(t)) & t \in (0, T), \\ v(0) = v(T). \end{cases} \tag{6.10}$$

Provided that

$$p, q \in (1, \infty) \quad \text{such that} \quad \frac{1}{p} + \frac{d}{2q} < 1 \tag{6.11}$$

the result on existence and uniqueness of the modified model reads as follows.

Theorem 6.3.2. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded C^3-domain, $\xi \in \mathbb{R}$ and assume that (6.11) is satisfied. Let $T > 0$ and $g = (g_1, g_2)^T \in \mathbb{F}$ be T-periodic and assume that the term τ is replaced by τ_{mod}.*

Then there is $r_0 > 0$ such that for any $r \in (0, r_0)$ there exists $\delta = \delta(r) > 0$ such that if $\|g\|_{\mathbb{F}} < \delta$, then there exists a T-periodic solution $v = (u, Q)^T \in \mathbb{E}$ to (BE)*, which is unique in $\overline{B_{\mathbb{E}}}(0, r)$.*

Proof. In order to prove the theorem, by Proposition 2.5.6 we need to show the following properties:

a) The operator $\mathcal{A}_{\mathrm{mod}}(0)$ admits maximal periodic L^p-regularity on X_0.

b) There exists $R_0 > 0$ such that for each $R \in (0, R_0)$ there exists $L(R) > 0$ such that

$$\|\mathcal{A}_{\mathrm{mod}}(v(\cdot))z(\cdot) - \mathcal{A}_{\mathrm{mod}}(\tilde{v}(\cdot))z(\cdot)\|_{\mathbb{F}} \leq L(R)\|v - \tilde{v}\|_{\mathbb{E}}\|z\|_{\mathbb{E}}$$

for all $v, \tilde{v}, z \in \overline{B_{\mathbb{E}}}(0, R)$.

c) $F_{\mathrm{mod}}(\cdot, v(\cdot)) \in \mathbb{F}$ for any $v \in \mathbb{E}$.

d) There exists a $C > 0$ such that for any $R > 0$ it is

$$\|F_{\mathrm{mod}}(\cdot, v(\cdot)) - F_{\mathrm{mod}}(\cdot, \tilde{v}(\cdot))\|_{\mathbb{F}} \leq CR\|v - \tilde{v}\|_{\mathbb{E}}$$

for all $v, \tilde{v} \in \overline{B_{\mathbb{E}}}(0, R)$.

Note that $\mathcal{A}_{\mathrm{mod}}(0)$ is given by

$$A_{\mathrm{mod}}(0) = \begin{pmatrix} A_D & 0 \\ \frac{2\xi}{n} D & D_N \end{pmatrix}.$$

Hence, we have a lower triangular structure. Since the operators A_D and D_N are invertible and admit maximal L^p-regularity on $L^q_\sigma(\Omega)$ and $W^{1,q}(\Omega; \mathbb{S}^d_{0,\mathbb{C}})$, respectively, the same holds true for the operator matrix $A_{\mathrm{mod}}(0)$ on X_0. Using Proposition 2.2.1 yields that $\mathcal{A}_{\mathrm{mod}}(0)$ admits maximal periodic L^p-regularity on X_0, which proves a).

By [84, Proposition 3.3.4] $\mathcal{A}_{\mathrm{mod}}$ is locally Lipschitz continuous provided p and q satisfy condition (6.11). This yields b).

For assertion c) and d), we proceed as in the proof of Theorem 6.2.1. By [84, Proposition 3.4.4] we obtain that $F_{\mathrm{mod}}(\cdot, v(\cdot)) \in \mathbb{F}$ is satisfied for $v \in \mathbb{E}$ and furthermore, for some $k \in \mathbb{N}_0$ we have

$$\begin{aligned}&\|F_{\mathrm{mod}}(\cdot, v(\cdot)) - F_{\mathrm{mod}}(\cdot, \tilde{v}(\cdot))\|_{\mathbb{F}} \\ &\quad \leq C(\|v\|_{L^\infty(0,T;X_\gamma)} + \|\tilde{v}\|_{L^\infty(0,T;X_\gamma)} + 1)^k \cdot (\|v\|_{\mathbb{E}} + \|\tilde{v}\|_{\mathbb{E}})\|v - \tilde{v}\|_{\mathbb{E}},\end{aligned}$$

for all $v, \overline{v} \in \mathbb{E}$. Using the same reasoning as above, this proves d) and hence, the proof is complete. $\square$

Bibliography

[1] H. Abels, G. Dolzmann, and Y. Liu. *Strong solutions for the Beris-Edwards model for nematic liquid crystals with homogeneous Dirichlet boundary conditions.* Advances in Differential Equations, **21** (2016), pp. 109–152.

[2] R. A. Adams and J. J. F. Fournier. Sobolev Spaces. Pure and Applied Mathematics, vol. 140, Elsevier/Academic Press, Amsterdam, 2003.

[3] R. R. Aliev and A. V. Panfilov. *A Simple Two-variable Model of Cardiac Excitation.* Chaos, Solitons & Fractals **7** (1996), no. 3, 293–301.

[4] H. Amann. Linear and quasilinear parabolic problems. Monographs in Mathematics, vol. 89, Birkhäuser, Boston, 1995.

[5] H. Amann. *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems.* In: Function Spaces, Differential Operators and Nonlinear Analysis, H. Schmeisser, H. Triebel (Eds.), Teubner, Stuttgart, 1993, 9–126.

[6] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. Vector-valued Laplace Transforms and Cauchy Problems. Monographs in Mathematics, vol. 96, Birkhäuser, Basel-Boston-Berlin, 2001.

[7] W. Arendt and S. Bu. *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity.* Math. Z. **240** (2002), no. 2, 311–343.

[8] H. Bahouri, J. Y. Chemin, and R. Danchin. Fourier Analy-

sis and Nonlinear Partial Differential Equations. Grundlehren der Mathematischen Wissenschaften, vol. 343, Springer, Heidelberg, 2011.

[9] M. Bendahmane and K. H. Karlsen. *Stochastically forced cardiac bidomain model.* Stochastic Processes and their Applications **129** (2019), no. 12, 5312–5363.

[10] J. Bergh and J. Löfström. Interpolation Spaces. An Introduction. Grundlehren der Mathematischen Wissenschaften, no. 223, Springer, Berlin, 1976.

[11] A. N. Beris and B. J. Edwards. Thermodynamics of flowing systems: with internal microstructure. Oxford Engineering Science Series, 1994.

[12] T. Black, J. Lankeit, and M. Mizukami. *Singular sensitivity in a Keller–Segel-fluid system.* J. Evol. Equ. **18** (2018), 561–581.

[13] Y. Bourgault, Y. Coudière, and C. Pierre. *Existence and uniqueness of the solution for the bidomain model used in cardiac electrophysiology.* Nonlinear Anal. Real World Appl. **10** (2009), no. 1, 458–482.

[14] L. Brandolese and M. Schonbek. *Large time behaviour of the Navier–Stokes flow.* In: Giga, Y., Novotný, A. (eds) Handbook of Mathematical Analysis in Mechanics of Viscous Fluids. Springer, Cham, 2018, 579–645.

[15] C. Cavaterra, E. Rocca, H. Wu, and X. Xu. *Global strong solutions of the full Navier–Stokes and Q-tensor system for nematic liquid crystal flows in two dimensions.* SIAM Journal on Mathematical Analysis, **48** (2016), no. 2, pp. 1368–1399.

[16] P. Colli Franzone, L. Guerri, and S. Tentoni. *Mathematical modeling of the excitation process in myocardial tissue: influence of fiber rotation on wavefront propagation and potential field.* Math. Biosci. **101** (1990), no. 2, 155–235.

[17] P. Colli Franzone, L. F. Pavarino, and S. Scacchi. Mathematical cardiac electrophysiology. MS&A. Modeling, Simulation and Applications, vol. 13, Springer, Cham, 2014.

[18] P. Colli Franzone, M. Pennacchio, and G. Savaré. *Mul-*

tiscale modeling for the bioelectric activity of the heart. SIAM J. Math. Anal. **37** (2005), no. 4, 1333–1370.

[19] P. Colli Franzone and G. Savaré. *Degenerate evolution systems modeling the cardiac electric field at micro- and macroscopic level.* In: Lorenzi, A., Ruf, B. (eds) Evolution Equations, Semigroups and Functional Analysis. Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, Basel, 2002, 49–78.

[20] G. Da Prato and P. Grisvard. *Sommes d'opérateurs linéaires et équations difféntielles opérationelles.* J. Math. Pures Appl. **54** (1975), no. 3, 305–387.

[21] R. Denk, G. Dore, M. Hieber, J. Prüss, and A. Venni. *New thoughts on old results of R. T. Seeley.* Math. Ann. **328** (2004), no. 4, 545–583.

[22] R. Denk, M. Hieber, and J. Prüss. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. Memories of the American Mathematical Society, American Mathematical Society, vol. 788, 2003.

[23] R. Denk, M. Hieber, and J. Prüss. *Optimal* L^p-L^q*-estimates for parabolic boundary value problems with inhomogeneous data.* Math. Z. **257** (2007), no. 1, 193–224.

[24] K.-J. Engel and R. Nagel. One-parameter semigroups for linear evolution equations. Graduate Texts in Mathematics, vol. 194. Springer, New York, 2000.

[25] J. L. Ericksen. *Hydrostatic theory of liquid crystals.* Arch. Rational Mech. Anal, **9** (1962), pp. 371–378.

[26] R. Farwig, H. Kozono, and H. Sohr. *Stokes semigroups, strong, weak, and very weak solutions for general domains.* In: Giga, Y., Novotný, A. (eds) Handbook of Mathematical Analysis in Mechanics of Viscous Fluids. Springer, Cham, 2018, 419–459.

[27] R. FitzHugh. *Impulses and Physiological States in Theoretical Models of Nerve Membrane.* Biophys. J. **1** (1961), no. 6, 445–466.

[28] G. P. Galdi. An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems. Springer Monographs in Mathematics, 2nd edition, Springer, New York, 2011.

[29] G. P. Galdi, M. Hieber, and T. Kashiwabara. *Strong time-periodic solutions to the 3D primitive equations subject to arbitrary large forces.* Nonlinearity **30** (2017), no. 10, 3979–3992.

[30] G. P. Galdi and M. Kyed. *Time-periodic solutions to the Navier–Stokes equations.* In: Giga, Y., Novotný, A. (eds) Handbook of Mathematical Analysis in Mechanics of Viscous Fluids. Springer, Cham, 2018, 509–578.

[31] M. O. Gani and T. Ogawa. *Stability of periodic traveling waves in the Aliev–Panfilov reaction-diffusion system.* Commun. Nonlinear Sci. Numer. Simul. **33** (2016), 30–42.

[32] P. G. de Gennes and J. Prost. The physics of liquid crystals. International Series of Monographs On Physics, Clarendon Press, 1995.

[33] Y. Giga and N. Kajiwara. *On a resolvent estimate for bidomain operators and its applications.* J. Math. Anal. Appl. **459** (2018), no. 1, 528–555.

[34] Y. Giga, N. Kajiwara and K. Kress. *Strong time-periodic solutions to the bidomain equations with arbitrary large forces.* Nonlinear Anal. Real World Appl. **47** (2019), no. 1, 398–413.

[35] M. Haase. The Functional Calculus for Sectorial Operators. Operator Theory: Advances and Applications, vol. 169, Birkhäuser Verlag, Basel, 2006.

[36] M. Hieber. *Analysis of Viscous Fluid Flows: An Approach by Evolution Equations.* In: Galdi, G., Shibata, Y. (eds) Mathematical Analysis of the Navier-Stokes Equations. Lecture Notes in Mathematics, vol 2254. Springer, Cham, 2020, 1–146.

[37] M. Hieber, A. Hussein, and M. Saal. *Global strong well-posedness of the stochastic bidomain equations with FitzHugh-Nagumo transport*. Available at `https://arxiv.org/abs/2002.03960`.

[38] M. Hieber, N. Kajiwara, K. Kress and P. Tolksdorf. *The periodic version of the Da Prato–Grisvard theorem and applications to the bidomain equations with FitzHugh–Nagumo transport.* Ann. Mat. Pura Appl. (2020), in press. https://doi.org/10.1007/s10231-020-00975-6.

[39] M. Hieber and J. Prüss. *Bounded $\mathcal{H}^\infty$-calculus for a class of nonlocal operators: the bidomain operator in the L^q-setting.* Math. Ann., (2019), in press. http://dx.doi.org/10.1007/s00208-019-01916-2.

[40] M. Hieber, and J. Prüss. *Functional calculi for linear operators in vector valued L^p-spaces via the transference principle.* Adv. Differential Equations **3** (1998), no. 6, 847–876.

[41] M. Hieber, and J. Prüss. *Modeling and analysis of the Ericksen–Leslie equations for nematic liquid crystal flows.* In: Giga, Y., Novotný, A. (eds) Handbook of Mathematical Analyis in Mechanics of Viscous Fluids, Springer, Cham, 2018, 1075–1134.

[42] M. Hieber, and J. Prüss. *On the bidomain problem with FitzHugh–Nagumo transport.* Arch. Math. **111** (2018), no. 3, 313–327.

[43] M. Hieber and J. Saal. *The Stokes equation in the L^p-setting: well-posedness and regularity properties.* In: Giga, Y., Novotný, A. (eds) Handbook of Mathematical Analysis in Mechanics of Viscous Fluids. Springer, Cham, 2018, 117–206.

[44] M. Hieber and C. Stinner. *Strong time periodic solutions to Keller–Segel systems: An approach by the quasilinear Arendt–Bu theorem.* J. Differential Equations. **269** (2020), no. 2, 1636–1655.

[45] T. Hillen and K. J. Painter. *A user's guide to PDE models for chemotaxis.* J. Math. Biol. **58** (2009), 183–217.

[46] D. Horstmann. *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences I.* Jahresber. Deutsch. Math.-Verein **105** (2003), no. 3 103–165.

[47] D. Horstmann, H. Meinlschmidt, and J. Rehberg. *The full Keller–Segel model is well-posed on nonsmooth domains.* Nonlinearity **31** (2018), no. 4, 1560–1592.

[48] C. Jin. *Large time periodic solution to the coupled chemotaxis-Stokes model.* Math. Nachr. **290** (2017), no. 11–12, 1701–1715.

[49] C. Jin. *Large time periodic solutions to coupled chemotaxis-fluid models.* Z. Angew. Math. Phys. **68** (2017), Art. 137, 1–24.

[50] J. Keener and J. Sneyd. Mathematical physiology. Interdisci-

plinary Applied Mathematics, vol. 8, Springer-Verlag, New York, 1998.

[51] T. Kato. Perturbation theory for linear operators. Die Grundlehren der mathematischen Wissenschaften, vol. 132. Springer, Berlin, 1995.

[52] E. F. Keller and L. A. Segel. *Initiation of slime mold aggregation viewed as an instability.* J. Theor. Biol. **26** (1970), no. 3, 399–415.

[53] M. Köhne, J. Prüss, and M. Wilke. *On quasilinear parabolic evolution equations in L_p-spaces.* J. Evol. Equ. **10** (2010), no. 2, 443–463.

[54] H. Kozono, M. Miura, and Y. Sugiyama. *Existence and uniqueness theorem on mild solutions to the Keller-Segel system coupled with the Navier–Stokes fluid..* J. Funct. Anal. **270** (2016), no. 5, 1663–1683.

[55] K. Kunisch and M. Wagner. *Optimal control of the bidomain system (I): the monodomain approximation with the Rogers-McCulloch model.* Nonlinear Anal. Real World Appl. **13** (2012), no. 4, 1525–1550.

[56] K. Kunisch and M. Wagner. *Optimal control of the bidomain system (II): uniqueness and regularity theorems for weak solutions.* Ann. Mat. Pura Appl. **192** (2013), no. 6, 951–986.

[57] K. Kunisch and M. Wagner. *Optimal control of the bidomain system (III): Existence of minimizers and first-order optimality conditions.* ESAIM: Math. Model. Numer. Anal. **47** (2013), no. 4, 1077–1106.

[58] K. Kunisch and M. Wagner. *Optimal control of the bidomain system (IV): corrected proofs of the stability and regularity theorems.* Available at `https://arxiv.org/abs/1409.6904`.

[59] J. Lankeit and M. Winkler. *Facing Low Regularity in Chemotaxis Systems.* Jahresber. Dtsch. Math. Ver. **122** (2020), 35–64.

[60] J. LeCrone, J. Prüss, and M. Wilke. *On quasilinear parabolic evolution equations in weighted L^p-spaces II.* J. Evol. Equ. **14** (2014), no. 3, 509–533.

[61] F. M. LESLIE. *Some constitutive equations for liquid crystals.* Arch. Rational Mech. Anal, **28** (1968), no. 4, pp. 265–283.

[62] A. LORZ. *Coupled chemotaxis fluid model.* Math. Models Methods Appl. Sci. **20** (2010), no. 6, 987–1004.

[63] A. LORZ. *A coupled Keller–Segel–Stokes model: global existence for small initial data and blow-up delay.* Commun. Math. Sci. **10** (2012), no. 2, 555–574.

[64] A. LUNARDI. Analytic semigroups and optimal regularity in parabolic problems. Progress in Nonlinear Differential Equations and their Applications, vol. 16, Birkhäuser, Basel, 1995.

[65] A. LUNARDI. Interpolation theory, 2nd edition. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie). [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], vol. 16, Edizioni della Normale, Pisa, 2009.

[66] C. LUO and Y. RUDY. *A model in the ventricular cardiac action potential. Depolarization, repolarization, and their interaction.* Circ Res. **68** (1991), no. 6, 1501–1526.

[67] Y. Mori and H. Matano. *Stability of front solutions of the bidomain equation.* Comm. Pure Appl. Math. **69** (2016), no. 12, 2364–2426.

[68] M. PAICU and A. ZARNESCU. *Global existence and regularity for the full coupled Navier-Stokes and Q-tensor system.* SIAM Journal on Mathematical Analysis, **43** (2011), no. 5, pp. 2009–2049.

[69] M. PAICU and A. ZARNESCU. *Energy dissipation and regularity for a coupled Navier–Stokes and Q-tensor system.* Arch. Rational Mech. Anal, **203** (2012), no. 1, pp. 45–67.

[70] J. PRÜSS and G. SIMONETT. *Maximal regularity for evolution equations in weighted L_p-spaces.* Arch. Math. **82** (2004), no. 5, 415–431.

[71] J. PRÜSS and G. SIMONETT. Moving Interfaces and Quasilinear Parabolic Evolution Equations. Monographs in Mathematics, vol. 105, Birkhäuser, Basel, 2016.

[72] J. PRÜSS, G. SIMONETT, and M. WILKE. *Critical spaces for quasilinear parabolic evolution equations and applications.* J. Differential Equations **264** (2018), no. 3, 2028–2074.

[73] J. PRÜSS, G. SIMONETT, and R. ZACHER. *On convergence of solu-*

tions to equilibria for quasilinear parabolic problems. J. Differential Equations **246** (2009), no. 10, 3902–3931.

[74] J. M. Rogers and A. D. McCulloch. *A collocation-Galerkin finite element model of cardiac action potential propagation.* IEEE Trans. Biomed. Eng. **41** (1994), no. 8, 743–757.

[75] E. M. Stein. Singular integrals and differentiability properties of functions. Princeton University Press, Princeton, 1986.

[76] A. Stevens. *Mathematics and the Life-Sciences: A Personal Point of View.* Jahresber. Dtsch. Math.-Ver. **119** (2017), no. 3, 143–168.

[77] H. Triebel. Interpolation Theory, Function Spaces, Differential Operators. North-Holland Mathematical Library, vol. 18, North-Holland Publishing, Amsterdam, 1978.

[78] L. Tung. *A bidomain model for describing ischemic myocardial d-c potentials.* PhD Thesis, MIT, 1978.

[79] I. Tuval, L. Cisneros, C. Dombrowski, C. W. Wolgemuth, J. O. Kessler, and R. E. Golstein. *Bacterial swimming and oxygen transport near contact lines.* Proc. Natl. Acad. Sci. USA **102** (2005), no. 7, 2277-2282.

[80] M. Veneroni. *Reaction-diffusion systems for the macroscopic bidomain model of the cardiac electric field.* Nonlinear Anal. Real World Appl. **10** (2009), no. 2, 849–868.

[81] M. Wilkinson. *Strictly physical global weak solutions of a Navier–Stokes Q-tensor system with singular potential.* Arch. Rational Mech. Anal., **218** (2015), no. 1, pp. 487–526.

[82] M. Winkler. *Global large-data solutions in a chemotaxis-(Navier–)Stokes system modeling cellular swimming in fluid drops.* Comm. Partial Differential Equations **37** (2012), no. 2, 319–351.

[83] M. Winkler. *Global weak solutions in a three-dimensional chemotaxis-Navier–Stokes system.* Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), no. 5, 1329–1352.

[84] M. Wrona. *Liquid Crystals and the Primitive Equations: An Approach by Maximal Regularity.* PhD Thesis, Technische Universität Darmstadt, 2020, http://tuprints.ulb.tu-darmstadt.de/11551/.

[85] I. Wood. *Maximal L^p -regularity for the Laplacian on Lipschitz domains.* Math. Z. **255** (2007), no. 4, 855–875.

[86] K. Yosida. Functional analysis. Die Grundlehren der Mathematischen Wissenschaften, vol. 123, Springer, Berlin–New York, 1980.

[87] D. Z. Zanger. *The Inhomogeneous Neumann Problem in Lipschitz Domains.* Comm. PDE **25** (2000), no. 9-10, 1771–1808.

List of notations

Sets

Spaces

$L^p_{av}(\Omega)$	L^p-integrable vector fields with vanishing mean	6
$L^p_{loc}(\Omega)$	locally L^p-integrable vector fields	6
$L^p_\sigma(\Omega)$	solenoidal L^p-integrable vector fields	21
$L^p_\mu(J;X)$	time-weighted L^p-space	23
$\mathcal{S}(\mathbb{R}^d;X)$	space of X-valued Schwartz functions	6
$\mathcal{S}'(\mathbb{R}^d;X)$	space of X-valued tempered distributions	6
V	space of all $H^1(\Omega)$-functions	82
$W^{k,p}(\Omega;X)$	X-valued Sobolev space	6
$W^{1,p}_0(\Omega)$	Sobolev space with Dirichlet boundary condition	8
$W^{k,p}_N(\Omega)$	Sobolev space with Neumann boundary condition	8
$W^{-1,q}(\Omega)$	dual space of $W^{1,q'}(\Omega)$	19
X'	toplogical dual space	4
$[X_0,X_1]_\theta$	complex interpolation space between X_0 and X_1	10
$(X_0,X_1)_{\theta,p}$	real interpolation space between X_0 and X_1	11
$X_{\gamma,\mu}$	real interpolation space $(X_0,X_1)_{\mu-1/p,p}$	23

Functions and operators

$\overline{z}$	complex conjugate of a complex number $z \in \mathbb{C}$	3
${}_{X'}\langle\cdot,\cdot\rangle_X$	dual pairing between X and X'	4
Δ_N	Neumann-Laplacian on $L^q(\Omega)$	17
$\Delta_{N,w}$	Neumann-Laplacian on $W^{-1,q}(\Omega)$	18
Δ^1_N	Neumann-Laplacian on $W^{1,q}(\Omega)$	18
D_N	shifted Neumann-Laplacian on $W^{1,q}(\Omega)$	143
A_D	Stokes operator on $L^q_\sigma(\Omega)$	21

Index

Curriculum Vitæ

10/2011 – 09/2014	**Bachelor's studies**, *Technische Universität Darmstadt*, Darmstadt, Germany, Bachelor of Science Mathematics Bachelor's thesis: *Der Satz von Alaoglu und schwache Topologien*
10/2014 – 09/2016	**Master's studies**, *Technische Universität Darmstadt*, Darmstadt, Germany, Master of Science Mathematics Master's thesis: *Periodic and almost periodic solutions to evolution equations*
10/2016 – 08/2018	**Doctoral scholarship of IRTG 1529 – Mathematical Fluid Dynamics,**, financial and academic sponsoring
10/2017 – 04/2018	**Research stay**, *The University of Tokyo*, Tokyo, Japan
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03/06/2020	**Submission of the doctor's thesis (Dissertation)** *Time-Periodic Solutions to Bidomain, Chemotaxis-Fluid, and Q-Tensor Models* at *Technische Universität Darmstadt*, Darmstadt, Germany
13/07/2020	**Defense of the doctor's thesis**, overall assessment *magna cum laude / sehr gut bestanden*